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Preload-adjustable compliant torsional quasi-zero-stiffness isolator for flexible beam structures

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ABSTRACT

With higher aspect ratios and the trend toward compliant designs in morphing aircraft, such structures are prone to undesired structural responses under fuselage vibration and aerodynamic loading. To address this issue, this paper proposes a preload-adjustable compliant torsional quasi-zero-stiffness (CT-QZS) isolator for low-frequency vibration isolation of flexible beam structures through boundary modification. The CT-QZS isolator integrates a negative-stiffness (NS) component composed of rotated bistable buckled beams and a positive-stiffness (PS) component composed of compliant curved beams. A tunable sliding groove on the rigid outer ring of the PS component allows continuous preload adjustment by changing the relative angle between the PS and NS components without altering the quasi-zero-stiffness (QZS) range. The torque-angle characteristic of the PS component is derived using Euler beam theory and the unit-load method and validated through quasi-static experiments. The threshold condition and parametric tuning rule of the negative stiffness are investigated through bifurcation analysis and parametric finite element simulations. Two CT-QZS isolators obtained from two different design strategies are integrated as non-invasive rotational boundaries at the root of a PLA cantilever beam. Experimental results demonstrate that the proposed preload-adjustable QZS isolators significantly reduce the first natural frequency and the vibration isolation onset frequency, achieving high static stiffness and low dynamic stiffness simultaneously compared to both the baseline fixed support and various linear isolators. The design offers an effective solution for low-frequency vibration isolation of flexible aerospace structures, such as high-aspect-ratio wings, morphing wings, and deployable space mechanisms.

1. Introduction

Lightweight beam-like structures are widely used in aerospace systems, including high-aspect-ratio wings, morphing aircraft and deployable space structures. These include high-aspect-ratio wings of modern civil aircraft [1], morphing wingtip devices for gust load alleviation [2], compliant mechanisms of morphing wings [3,4], and large space deployable structures [5]. Higher aerodynamic efficiency and lower structural mass require larger aspect ratios and lower structural stiffness. This often reduces natural frequencies and increases susceptibility to structural vibration [1]. Morphing wings using compliant mechanisms are also prone to undesired vibrations

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under aerodynamic loads, which can cause dynamic instability [3]. Space deployable structures, such as solar panels, frequently exhibit low-frequency vibrations following deployment. In space, where aerodynamic damping is absent, such vibrations may persist for long periods and disturb sensitive onboard instruments [6,7]. In many cases, such systems can be idealized as cantilever beams with elastically restrained roots. These boundary conditions can be represented by rotational springs, with the limiting cases corresponding to clamped and hinged supports. Undesired vibrations may propagate between different components through these elastic couplings. An effective mitigation strategy is to replace the elastic connection with a torsional vibration isolator, thereby suppressing the transmission of vibrations from the flexible structure to the base.

Linear vibration isolators mainly take two forms: mechanical and compliant. Mechanical linear rotational isolators typically use helical springs, torsion bars, or cam-roller mechanisms. Compliant designs rely on elastic deformation of curved beams or flexure pivots [8,9]. Quasi-zero-stiffness (QZS) isolators exhibit force-displacement characteristics similar to those of constant-force mechanisms (CFMs). The distinction is mainly in their use: QZS isolators are designed for dynamic isolation, whereas CFMs are typically used in quasi-static applications [8]. Two main approaches are used to achieve zero stiffness: the stiffness combination method and the pre-shaped curved beam method.

The stiffness combination method has been widely applied in translational QZS isolators. The classical three-spring configuration [10,11] uses horizontal springs to generate negative stiffness, compensating for the positive stiffness of vertical springs. However, the QZS region is narrow and sensitive to load variations [12]. To enlarge the low-stiffness range, researchers have introduced multiple pairs of oblique springs [13], X-shaped or scissor-like structures [14,15], and cam-roller-spring mechanisms [12,16]. Euler buckled beams [17,18], sliding beams [19], and magnetic springs [20,21] have also been used as negative-stiffness elements. Shaw et al. [22] proposed an innovative design that uses the transverse buckling of a bistable composite plate to provide negative stiffness, which is then connected in parallel with a linear spring to form a high-static-low-dynamic-stiffness (HSLDS) isolator. Experiments showed that this design significantly reduces the resonance frequency and peak response. Furthermore, viscoelastic dampers can significantly improve isolation efficiency at high frequencies, causing the transmissibility to decay more rapidly with increasing frequency [23]. Nonlinear damping and fractional derivative models can describe actual damping characteristics more accurately [24,25].

The pre-shaped curved beam method tailors the initial geometry of a single elastic element to exhibit QZS behaviour during deformation. This approach offers structural integration, no friction, and easy miniaturization. However, its load capacity is usually lower than that of the stiffness combination method. QZS characteristics have been achieved using sinusoidal beams [26] and semicircular arch assemblies [27], straight-arch-straight tandem beams [28], and NURBS shell-based metamaterials [29,30]. In addition to the above structures, Ye and Ji [31] proposed a truss-spring-based stacked Miura-ori structure. Linear springs replace traditional creases, achieving QZS through geometric nonlinearity and variable Poisson's ratio. Bio-inspired structures [32,33] also show strong nonlinear stiffness tuning capability. Broadband isolation can be achieved using periodic QZS units [21,27,34] or by introducing viscoelasticity [35].

Torsional vibration isolators can also be classified as mechanical or compliant [9]. Mechanical torsional QZS isolators have been realized using cam-roller mechanisms [36], torsional magnetic springs with negative stiffness [37,38], and rack-and-pinion systems [39]. To overcome the limitation of unidirectional negative stiffness, Zhang et al. [40] proposed a bidirectional torsional negative stiffness (BTNS) mechanism which converts decreasing spring force into increasing torque through geometric reconfiguration. This achieves bidirectional energy balancing and reduces the required driving torque by over 90%. Compliant torsional QZS isolators often use pre-buckled beams or pre-shaped curved beams [9]. Hou and Lan [41] achieved constant torque output over a range of 20°–90° by optimizing the shape of distributed-compliance limbs. Seyedkanani and Akbarzadeh [42] proposed a magnetically assisted multistable metamaterial that generates negative torsional stiffness and dissipates energy through snap-back instabilities. Noh et al. [43] proposed a torque-dependent stiffness-softening mechanism which reduces stiffness by shortening the moment arm of tension springs during rotation.

Most existing isolators focus on rigid-body motion with finite degrees of freedom. Few studies have considered continuous flexible structures, whose dynamics involve distributed degrees of freedom. For continuous beams, the tuned mass damper (TMD) [44] and nonlinear energy sink (NES) [45] are widely used. Both require structural modification. A TMD suppresses vibration by attaching a mass-spring-damper system tuned to a target frequency [46,47]. However, the TMD has a narrow effective bandwidth and is sensitive to changes in the host beam's parameters. When the beam's stiffness degrades due to damage or environmental factors, the vibration suppression performance deteriorates significantly [46]. NESs use strong nonlinearity and do not require tuning. They rely on targeted energy transfer (TET) for broadband energy dissipation [46,48]. Ahmadabadi and Khadem [49] attached a cubic NES to the free end of a cantilever beam and investigated its impact vibration suppression effect. Kani et al. [50] studied parameter optimisation and modal interactions. However, NES performance depends on an energy threshold and still requires added mass and nonlinear elements [51].

Advanced NES designs include vibro-impact NES (VI-NES) [52], bistable NES (BNES) [53], and magnetic tristable NES [54]. Fang et al. [52] showed that VI-NES induces both TET and inter-modal energy scattering (IMES). Energy is transferred from low to high modes, enhancing dissipation. Distributing multiple lightweight NES units along a beam can suppress multi-mode vibrations [54]. Another approach is to convert the beam into a metamaterial by embedding QZS resonators [47,55]. These resonators create ultra-low-frequency bandgaps and block wave propagation without large added mass [47].

Whether using TMDs or NESs, integration and modification of the existing structure are required, which is undesirable for integration into existing lightweight aerospace structures. In contrast, using boundary conditions to achieve vibration isolation is a non-invasive modification: it only requires changing the connection method, and the structural natural frequency can be increased or decreased by adjusting the boundary stiffness. Wang et al. [56] placed an intermediate elastic support at a mode node from the perspective of frequency maximization, using a finite stiffness to raise the natural frequency to that of the next higher mode. Alternatively, the structural natural frequency can be lowered to achieve ultra-low frequency vibration isolation. Tian et al. [57] introduced

a torsional QZS isolator at the free end of a beam, using a linear spring connected in parallel with a torsional spring to realize high-static-low-dynamic stiffness, reducing the first resonance frequency to nearly 0 Hz. Furthermore, Tian et al. [58] adopted a three-spring HSLDS isolator for a rectangular plate supported at its four corners, connecting a vertical spring in parallel with oblique springs, which reduced the low-frequency peak by over 80%. Zhang et al. [59] compared linear isolation, nonlinearly enhanced isolation, and QZS isolation. By adjusting a horizontal spring to generate negative linear stiffness that cancels the positive stiffness of a vertical spring, they demonstrated that QZS performs best in the low-frequency range. In summary, boundary-based QZS isolation enables flexible frequency tuning, effective low-frequency isolation, and easy integration.

Existing torsional QZS isolators still face three main limitations when applied to flexible beam structures. First, the preload capacity and the QZS range are often coupled, making it difficult to accommodate different static loads after fabrication. Second, many torsional QZS designs rely on complex multi-component mechanisms, such as cams, rollers, springs, or magnetic elements, which are sensitive to manufacturing and assembly errors. Third, for continuous flexible structures such as beams and plates, common vibration suppression methods, including tuned mass dampers and nonlinear energy sinks, usually require additional attachments or modifications to the host structure. In contrast, modifying the support boundary provides a non-invasive route for low-frequency vibration isolation, but preload-adjustable compliant torsional QZS boundaries for flexible structures remain insufficiently explored.

To address these issues, this paper proposes a preload-adjustable compliant torsional quasi-zero-stiffness (CT-QZS) isolator for flexible beam structures. The design adopts a stiffness-combination strategy, in which a negative-stiffness component composed of rotated bistable buckled beams is connected in parallel with a positive-stiffness component composed of compliant curved beams. A tunable sliding groove is introduced to adjust the relative assembly angle between the positive and negative stiffness components, allowing the preload torque to be tuned while maintaining the QZS operating range. The isolator is fabricated by fused deposition modelling (FDM) and assembled using detachable mortise-and-tenon joints. The mechanical characteristics are investigated through theoretical modelling, bifurcation analysis, finite element simulations and quasi-static experiments. Finally, the isolator is implemented as a non-invasive rotational boundary at the root of a cantilever beam, and its vibration isolation performance is evaluated through modal, swept-sine and random vibration tests.

The remainder of this paper is organized as follows. Section 2 presents the isolator design and integration. Section 3 describes fabrication and experiments. Section 4 develops the mechanical models and QZS design strategy. Section 5 evaluates vibration isolation performance. Section 6 discusses the potential application and limitations, and Section 7 concludes the paper.

2. Design scheme of the torsional CT-QZS vibration isolator

This section introduces the structural design and integration scheme of the proposed CT-QZS isolator. The composition of the CT-QZS isolator is first described, including the negative-stiffness component, positive-stiffness component and bearing support. The integration of the module with the flexible beam root is then presented to clarify how the isolator acts as a rotational boundary.

2.1. Compliant torsional quasi-zero-stiffness module

The CT-QZS module is shown in Fig. 1. The module consists of an NS component, a bearing support component, and a PS component. The NS component is composed of angle-adjustable rigid outer rings, compressed buckled beams, rigid inter rings, and connecting nuts. The bearing support component consists of a square shaft adapter sleeve, a bearing, and bearing support parts. The PS component is composed of a fixed outer ring, circular arc curved beams, an inner ring for connecting the NS component, and

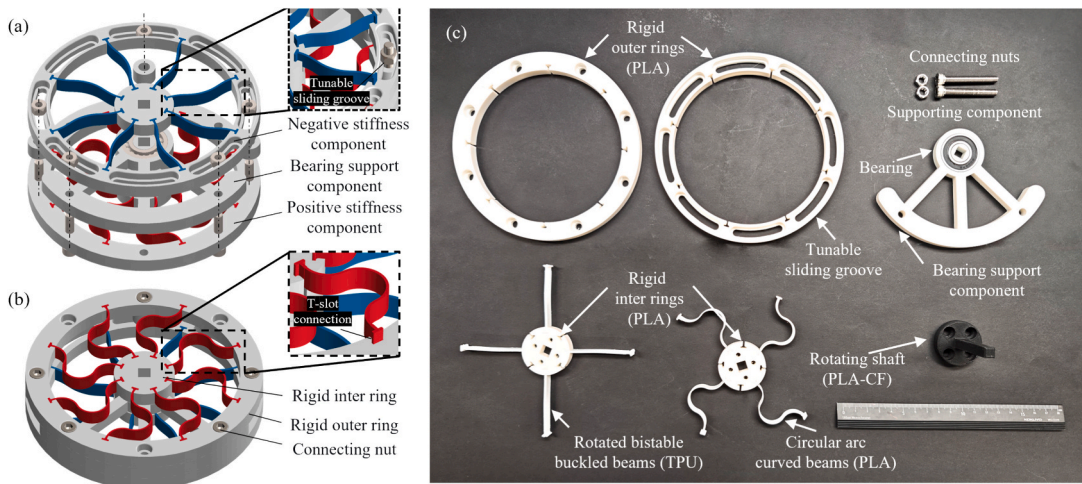


Fig. 1. Compliant torsional quasi-zero-stiffness module. (a) Exploded view of the module; (b) Positive stiffness component: circular arc curved beam; (c) Negative stiffness component: rotated pre-compressed beam.

connecting screws.

Both ends of the compressed buckled beams and the circular arc curved beams are designed with T-slot connectors. One end of each beam is inserted into the T-shaped groove on the outer side of the adjustable-angle outer ring and the fixed outer ring, while the other end is inserted into the inner ring for connecting the NS component and the inner ring for connecting the PS component. In the CT-QZS module, the NS and PS components mainly carry torsional loads, while the bearing support component mainly carries radial shear loads. This detachable T-slot connection design allows the internal flexible beams of the isolator to be replaced directly when the system is subjected to different load conditions. For small load variations, tunable sliding grooves are integrated into the rigid outer rings of the NS component, as shown in the inset of Fig. 1(a). This enables continuous adjustment of the load within a certain range.

All rigid inter-rings and the square shaft adapter sleeve are rotationally constrained to the rotating shaft through square holes. This configuration connects the NS and PS components in parallel and achieves the desired QZS behaviour. At the layer of the bearing support components, inter-layer support rings support both the fixed outer ring and the adjustable-angle outer ring. Screws pass through the holes of the fixed outer ring, the bearing support components, the inter-layer support rings, and the adjustable-angle outer ring, fastening the components across different layers of the module. The assembly video of the system is provided in Supplementary Video S1.

2.2. Compliant torsional QZS vibration isolator for flexible beam structures

As shown in Fig. 2, the beam structure consists of a support base, a clamping plate, screws, and a flexible beam structure. Square shafts are designed on both sides of the clamping support. These shafts are connected through the inner ring of the NS component, the square shaft adapter sleeve, and the inner ring of the PS component. This configuration couples the torsional degrees of freedom of the NS component and the PS component of the QZS module with the root of the flexible beam structure. Several through holes are designed in the cantilever beam clamping support, the clamping plate, and the flexible beam structure, and clamping screws are used to fully fasten all parts of the flexible beam structures.

The QZS module is fully fastened to the fixing nuts on the support base of the cantilever structure using screws. This constrains the rotational degrees of freedom on the outer sides of both the fixed outer ring and the adjustable-angle outer ring. The adjustable-angle outer ring has several stepped sliding grooves evenly distributed along its circumference. Each groove is sized to recess the connecting nut and constrain its rotation, while the through-groove allows the screw to pass through. When adjusting the relative angle θ_0 , the preload in the screws is reduced. This removes friction between the connecting nut and the stepped sliding groove. The adjustable-angle outer ring is then rotated manually to the desired angle θ_0 relative to the fixed outer ring. After adjustment, the screws are tightened to apply preload. The resulting friction fully constrains the adjustable-angle outer ring. To increase friction, anti-slip patterns may be added to the contact surfaces between the adjustable-angle outer ring and the support base, and between the adjustable-angle outer ring and the connecting nut. Several stepped holes are designed at the bottom of the support base to allow attachment to the target structure.

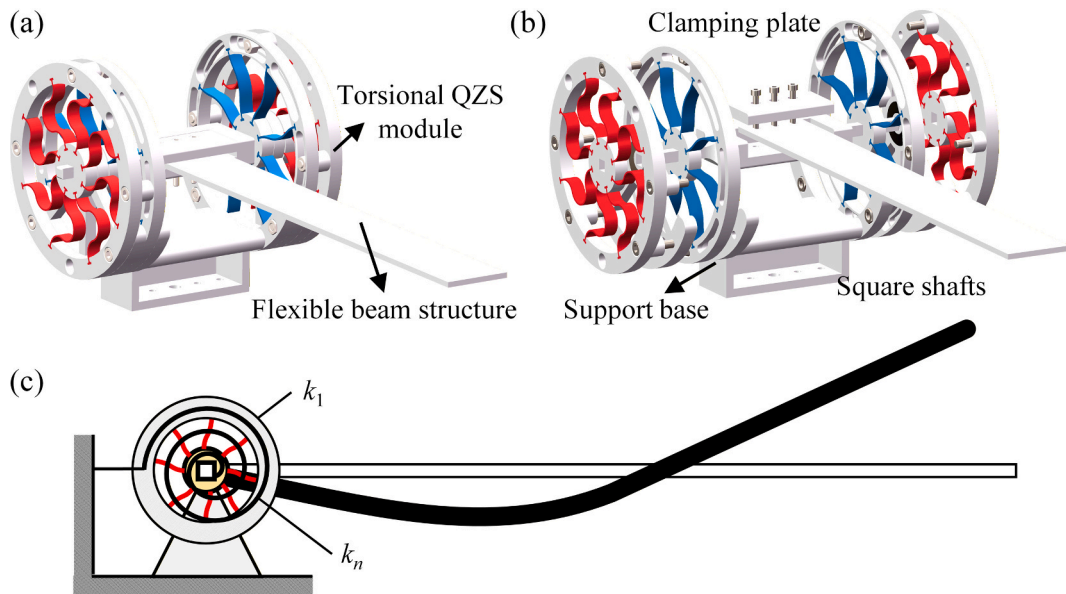


Fig. 2. CT-QZS vibration isolator for flexible beam structures (a) Overall assembly view; (b) Exploded view of the module; (c) Simplified schematic diagram of the system.

3. Manufacturing details and quasi-static torque-angle test setup

This section describes the fabrication method, material characterization and quasi-static torque-angle test setup used for component-level validation. The printed materials and process parameters are first introduced, followed by tensile tests for obtaining the material properties used in the modelling. The quasi-static test platform is then described, which is used to measure the torque-angle characteristics of the PS, NS and assembled CT-QZS components.

3.1. Additive manufacturing and material characterization

All test specimens were fabricated using a Bambu H2D FDM 3D printer. Fig. 1(c) shows photographs of all specimens, with the materials of the main components indicated. The NS component, which requires large compliant deformation, was printed using Bambu TPU 95A HF (1.75 mm diameter). This is a thermoplastic elastomer with high strain capacity. The main structural components and the PS component were printed using Bambu PLA Basic filament (1.75 mm diameter). All parts were modeled in SolidWorks and sliced using Bambu Studio. The layer height was set to 0.08 mm. A 100% infill with a rectilinear pattern was used.

To obtain accurate material parameters, three TPU 95A specimens and three PLA specimens were fabricated using the same printing parameters as the experimental modules. Uniaxial tensile tests were conducted using a universal testing machine (UTM4503, SUNS) according to ASTM D412 and ASTM D638, respectively. The dimensions of the standard specimens are shown in the inset of Fig. 3. The results are presented in Fig. 3.

Because TPU 95A shows pronounced hyperelastic behaviour, linear fitting was performed on the strain range below 0.1. The fitted modulus was 12.429 MPa, as shown in Fig. 3(a). To assess the validity of this simplification, a third-order Ogden hyperelastic model was implemented in ABAQUS and compared with the linear elastic model based on the fitted modulus, as detailed in Appendix A. The comparison shows that the linear elastic approximation is adequate for the present analysis and facilitates the subsequent theoretical derivations.

For PLA, the elastic modulus was obtained by linearly fitting the initial linear regions of the three tested specimens, giving a value of 1.840 GPa, as shown in Fig. 3(b). The printed PLA specimens departed from the initial linear elastic regime at a strain of approximately 0.018 and fractured at a strain of approximately 0.04. For structures subjected to repeated snap-through deformation, such irreversible plastic deformation and fracture are undesirable. Therefore, the NS component was printed using TPU, as the rotated bistable buckled beams require repeated large-deformation snap-through with recoverable strain.

3.2. Quasi-static experiment setup

To obtain the relationship between torque and rotation angle, a quasi-static experimental setup for torsional components was developed using LabVIEW, as shown in Fig. 4. A motion control card (NET6043-S2XE, HKTECH) was used to drive a 42-type stepper motor. The motor drove an electric rotary stage to realize angle-controlled rotation and record the rotation angle. A torque sensor (DYJN-104, Dayang) with a range of 5 Nm was used to measure the load torque in real time. The rotation angle and torque signals were synchronously acquired using the same motion control card. A LabVIEW-based host computer sent control commands via LAN communication. It also performed data acquisition, display, and storage.

4. Mechanical modelling and finite element validation

This section establishes the mechanical model for constructing the preload-adjustable CT-QZS response. The PS component is first

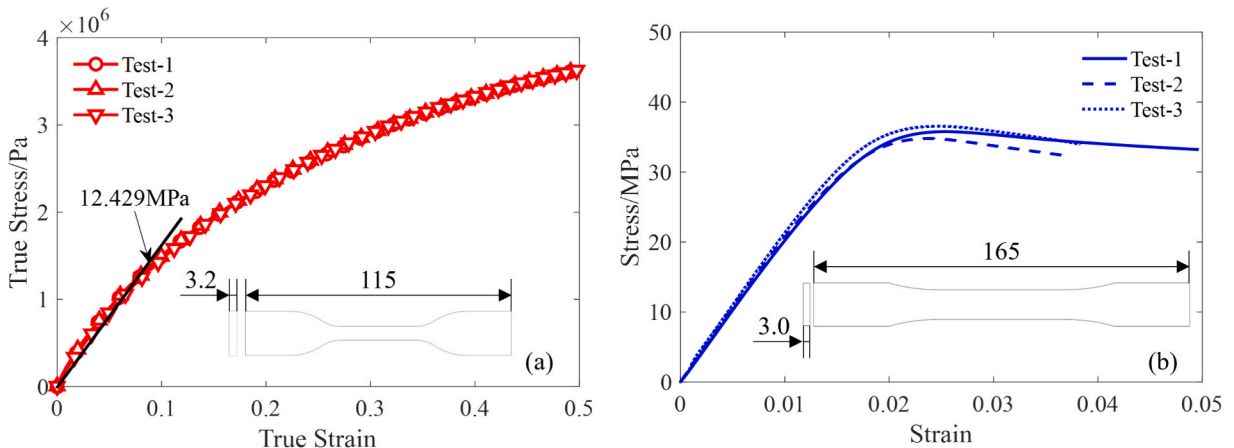


Fig. 3. Experimental results of standard material tests. (a) Uniaxial tensile test results of TPU 95A conducted according to ASTM D412; (b) Uniaxial tensile test results of PLA conducted according to ASTM D638.

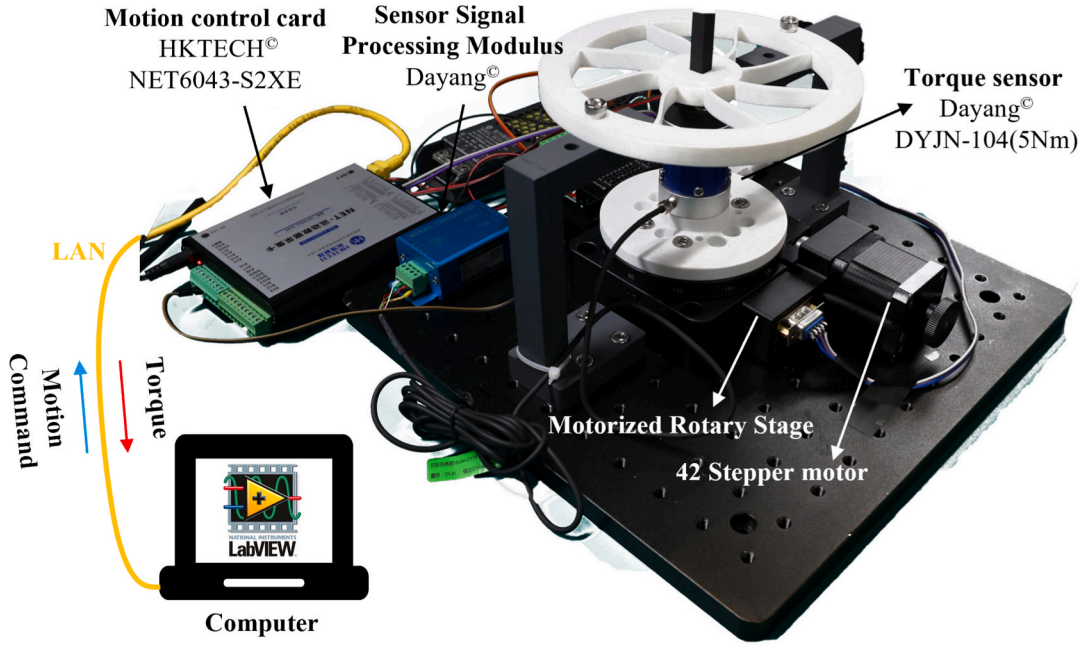


Fig. 4. Experimental setup for quasi-static experiments.

modelled using Euler beam theory and the unit-load method. The NS component is then analysed through bifurcation theory and finite element simulations, with additional mesh convergence to support the numerical model. Finally, the PS and NS components are combined to construct the CT-QZS characteristic, and the preload-adjustment mechanism based on the relative angular offset is derived.

4.1. Positive-stiffness component: Circular arc curved beams

The linear positive stiffness of the circular arc curved beams was theoretically derived using the unit-load method based on Mohr's integral. Following the coordinate system shown in Fig. 5(a), the coordinates (x_p, y_p) ($p = 1, 2, 3$) can be expressed as

$$x_1 = r + L/4 \sin \psi, y_1 = L/4(1 - \cos \psi) \psi \in [0, \pi/2] \quad (1)$$

$$x_2 = r + L/4(2 - \cos \psi), y_2 = L/4(1 + \sin \psi) \psi \in [0, \pi] \quad (2)$$

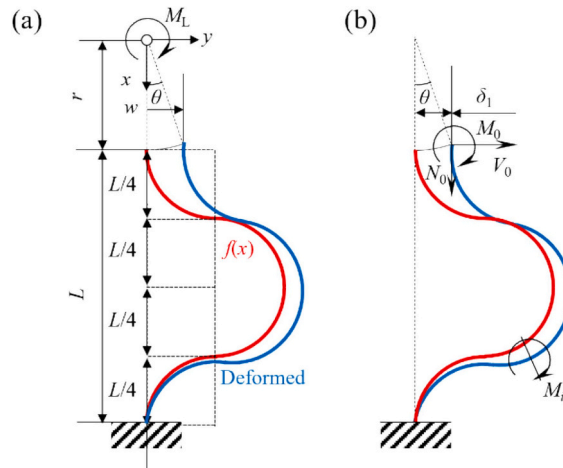


Fig. 5. Schematic of the deformed configuration for circular arc curved beams. (a) Definition of the coordinate system and geometric relationships of the circular arc curved beams. M_L is the driving torque. (b) Internal forces on different cross-sections of the curved beam.

$$x_3 = r + L/4(4 - \cos\psi), y_3 = L/4(1 - \sin\psi) \psi \in [0, \pi/2] \quad (3)$$

where ψ represents the parametric coordinate, and p denotes the index for different arc segments. The T-slot connection is rigid. Thus, one end of each flexible curved beam is clamped to the rigid outer rings. The other end connects to the rigid inter rings and acts as a rotational boundary with a given radius. Fig. 5(a) illustrates the simplified boundary conditions. L represents the difference in radius between the rigid outer rings and the rigid inter rings, i.e., the span of the curved beam along the x -direction; r and θ denote the rotational radius and rotation angle of the boundary, respectively; and w is the deflection of the curved beam along the y -direction.

As shown in Fig. 5(b), when an arbitrary bending moment M_0 , axial force N_0 , and shear force V_0 are applied at the root of the curved beam, the internal bending moment M_i along the cross-section of the curved beam is given by

$$M_i = -N_0 y_i + V_0 x_i + M_0 \quad (4)$$

According to the unit-load method, the structural displacement δ_j can be expressed as

$$\delta_j = \frac{L}{4EI} \int_0^{\pi/2} \left(M_1 \frac{\partial M_1}{\partial j} + M_3 \frac{\partial M_3}{\partial j} \right) d\psi + \frac{L}{4EI} \int_0^{\pi} M_2 \frac{\partial M_2}{\partial j} d\psi \quad j \in \{N_0, V_0, M_0\} \quad (5)$$

where the subscript j denotes the structural displacement generated by different types of unit loads, E and I are the Young's modulus and the cross-sectional moment of inertia of the material, respectively. Equation (5) can be written in matrix form as

$$\delta = \mathbf{C}\mathbf{F} \quad (6)$$

where $\mathbf{C}$ is the compliance matrix obtained from Eq. (5), $\delta = \{\delta_{N_0}, \delta_{V_0}, \delta_{M_0}\}^T$, and $\mathbf{F} = \{N_0, V_0, M_0\}^T$.

The rotational boundary condition with rotation radius at $(x = r, y = 0)$ can be written as

$$\delta_{BC}|_{x=r, y=0} = \{\theta, 0, -r\theta\}^T \quad (7)$$

Substituting Eq. (6) into Eq. (7) and taking the inverse yields the boundary load $\mathbf{F}_{BC}|_{x=r, y=0}$

$$\mathbf{F}_{BC}|_{x=r, y=0} = \mathbf{C}^{-1}\delta \quad (8)$$

The relationship between the driving torque M_L and the boundary load is

$$M_L = M_0 - rF_0 \quad (9)$$

Meanwhile, the driving torque can also be expressed as the product of the equivalent linear torsional stiffness K_P and the driving rotation angle

$$M_L = K_P \theta \quad (10)$$

Combining Eqs. (5)–(10), the expression for the equivalent linear torsional stiffness K_P in terms of the geometric parameters is obtained as

$$K_P = -\frac{2EI[(23\pi - 24)L^2 + 32\pi Lr + 32\pi r^2]}{\pi(5\pi - 8)L^3} \quad (11)$$

Furthermore, to normalize Eq. (11) with respect to the beam cross-section, both sides are divided by the flexural rigidity EI , yielding

$$\bar{K}_P = \frac{K_P}{EI} = -\frac{2[(23\pi - 24)L^2 + 32\pi Lr + 32\pi r^2]}{\pi(5\pi - 8)L^3} \quad (12)$$

To validate the derived formulas, beams with four different thicknesses t were fabricated and tested [Fig. 6(a)]. The circular arc curved beams were also analysed using the finite element software ABAQUS, with the analysis step defined as "Static, General". The beams were modeled using B31 elements. The element size was set to 1 mm. The force–displacement curves for beams with different thicknesses are shown in Fig. 7. The detailed theoretical, numerical and experimental results are listed in Table 1. It can be observed that the theoretical results agree well with the experiments for thinner beams. As the thickness increases, the error gradually increases.

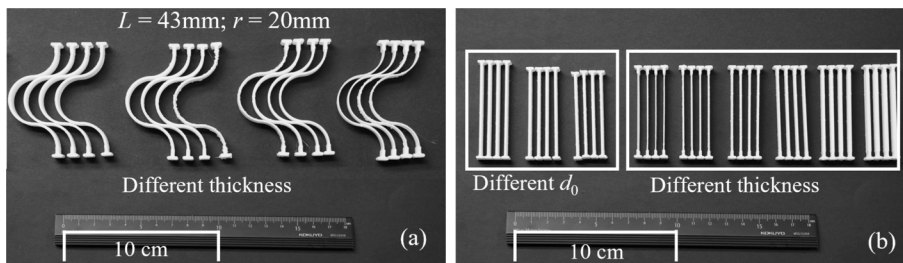


Fig. 6. Test specimens with different dimensions. (a) Circular arc curved beams; (b) Rotated bistable buckled beams.

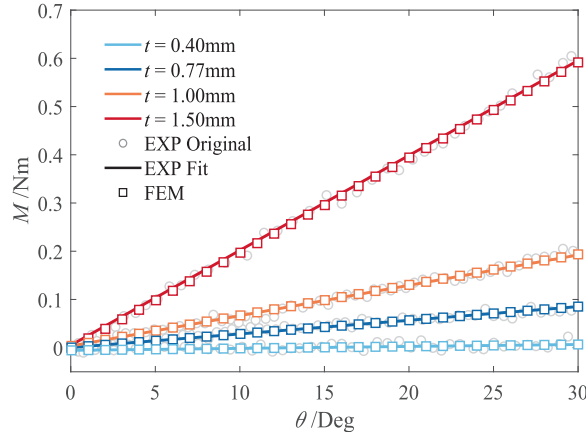


Fig. 7. Quasi-static loading experimental results of the circular arc curved beams.

Table 1

Comparison between theoretical, numerical (FEM) and experimental (EXP) results of circular arc curved beams.

Thickness/mm	K_p (EXP)/Nm	K_p (FEM)/Nm	K_p/EI (EXP)/m ⁻¹	K_p/EI (FEM)/m ⁻¹	K_p/EI (Theory)/m ⁻¹	Theory Error/%
1.50	1.131	1.172	202.44	226.47	224.274	9.736
1.00	0.3391	0.3473	221.13	226.45		1.424
0.77	0.1531	0.1585	218.65	226.38		2.509
0.40	0.02184	0.02222	222.45	226.39		0.818

The theoretical model tends to overestimate the stiffness. This is because Euler–Bernoulli beam theory neglects shear deformation, which leads to an overestimation of stiffness.

4.2. Negative-stiffness component: Rotated bistable buckled beams

4.2.1. Bifurcation analysis

According to the study by Radisson et al. [60], elastic strips exhibit violent snap-through instabilities under asymmetric loading. Bifurcation analysis has been widely used to identify the stable and unstable equilibrium branches that organize snap-through transitions in bistable and multistable beam systems [61]. In particular, when one end is clamped and the other end is rotated, saddle-node (SN) bifurcation occurs [62]. For the rotated bistable buckled beams in this study, the rotational boundary has a finite rotation radius [Fig.8(a)]. In previous studies, the rotating boundary condition was treated as having zero rotation radius. Therefore, the rotation radius must be further included in the theoretical model.

Based on Euler beam theory, the governing equation of the elastic beam is obtained as

$$EI \frac{\partial^4 w}{\partial x^4} + N \frac{\partial^2 w}{\partial x^2} = 0 \quad (13)$$

where N is the axial force and w is the beam deflection.

Since the axial deformation of the beam is relatively small and can be neglected, the inextensibility condition of the beam is introduced as a nonlinear constraint

$$\int_0^L \left(\frac{\partial w}{\partial x} \right)^2 dx = 2d_0 \quad (14)$$

where d_0 represents the axial compression distance of the beam.

To nondimensionalize the problem, Eqs. (13) and (14) are rewritten in the following dimensionless forms

$$\frac{\partial^4 W}{\partial X^4} + \Lambda^2 \frac{\partial^2 W}{\partial X^2} = 0 \quad (15)$$

$$\int_0^1 \left(\frac{\partial W}{\partial X} \right)^2 dX = 2 \quad (16)$$

where the dimensionless parameters are defined as

$$W = w/\sqrt{Ld_0}, \Lambda^2 = FL^2/BX = x/L, \Theta = \theta\sqrt{L/d_0} \quad (17)$$

Under this dimensionless framework, the boundary conditions are written as

$$W(1) = 0, \Theta(1) = 0$$

$$W(0) = \Theta r/L, \Theta(0) = \Theta \quad (18)$$

Equation (15) has the general solution of the form

$$W(X) = A \sin \Lambda X + B \cos \Lambda X + CX + D \quad (19)$$

The coefficients A, B, C, D in the general solution are determined by the boundary conditions in Eq. (18). Substituting Eq. (19) into (18) yields the following matrix form

$$\mathbf{M}\mathbf{v} = \mathbf{b} \quad (20)$$

where $\mathbf{v} = \{A, B, C, D\}^T$ and $\mathbf{b} = \{0, 0, \Theta r/L, \Theta\}^T$.

After substituting the solution of Eq. (20) into Eq. (19), the general solution is further inserted into the nonlinear constraint condition Eq. (16) to obtain a nonlinear equation for the dimensionless longitudinal compression Λ . From the boundary conditions in Eq. (18), it can be seen that the system depends only on the dimensionless rotation angle Θ and the ratio of the rotation radius to the beam length, r/L . Fig. 9 shows the bifurcation diagrams of the system for different r/L values, where the bifurcation variable is Θ . Fig. 9 (a) and 9(b) present the bifurcation diagrams of the dimensionless longitudinal compression Λ and the midpoint deflection $W(1/2)$, respectively. Since the system is symmetric, only the bifurcation parameter $\Theta > 0$ is considered.

As shown in Fig. 9(b), when $r/L = 0$, the model is consistent with the study by Radisson et al. [60], and the system exhibits a saddle-node bifurcation at the bifurcation parameter $\Theta \approx 1.782$, which agrees well with the experimental results of [62]. As r/L increases (with $r < L$), the saddle-node bifurcation structure remains. The bifurcation dimensionless angle first increases and then decreases. This means that, for a fixed L/d ratio, the rotation angle at the bifurcation also first increases and then decreases.

4.2.2. Finite element model and mesh convergence

The bifurcation analysis indicates that the rotated bistable buckled beam exhibits a strongly nonlinear snap-through response, which cannot be fully captured by the Euler beam-based analytical model. Therefore, finite element simulations were conducted in ABAQUS to obtain the post-buckling torque-angle characteristics. The beam was modelled using B31 beam elements with geometric nonlinearity enabled, and the material properties were taken from the experimental characterization in Fig. 3. The analysis was performed using the ‘‘Dynamic, Implicit’’ procedure under quasi-static loading conditions. Following the geometry and boundary conditions shown in Fig. 8 (b), one end of the beam was fixed, while the other end was first compressed by d_0 to generate the pre-buckled configuration and then rotated about the finite-radius end. A small initial angular perturbation was introduced to guide the beam toward a prescribed stable buckling branch. The reaction torque at the driven end was recorded as a function of the applied rotation angle to obtain the torque-angle response.

A mesh convergence study was conducted before the parametric simulations using a representative NS beam with $r = 20$ mm, $d_0 = 8$ mm, $L = 40$ mm, and a beam thickness of 1 mm. As shown in Fig. 8(c), the coarse meshes of 6 and 4 mm led to visible deviations and numerical oscillations near the snap-through region. As the element size is reduced, the predicted torque-angle response gradually converges. The curves obtained using 1, 0.5, and 0.25 mm meshes almost overlap over the full rotation range, including the local snap-

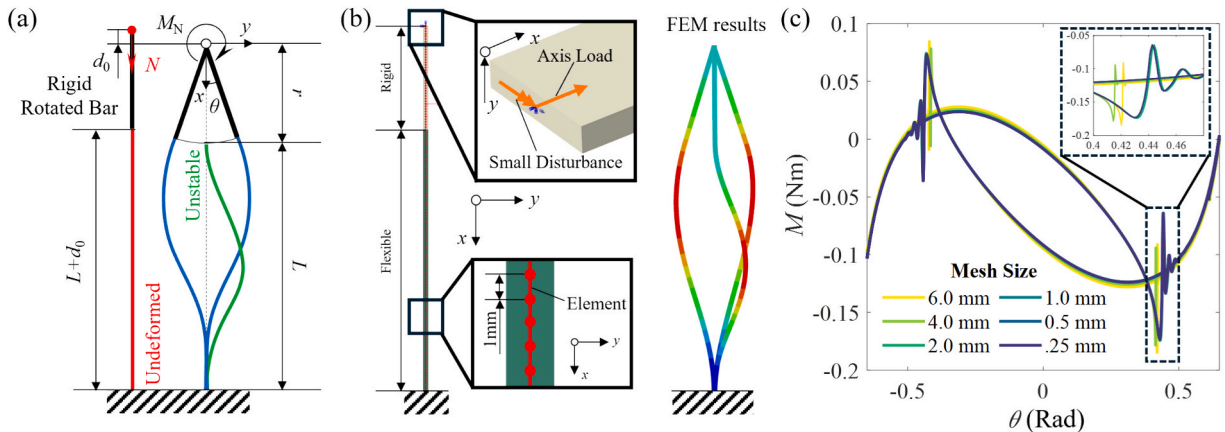


Fig. 8. Schematic of the deformed configuration for rotated bistable buckled beams. (a) Definition of the coordinate system and geometric relationships of the rotated bistable buckled beam. M_N is the driving torque. (b) Finite element model and deformation results. (c) Mesh convergence study for a representative NS beam. The inset in (c) shows a magnified view of the snap-through region.

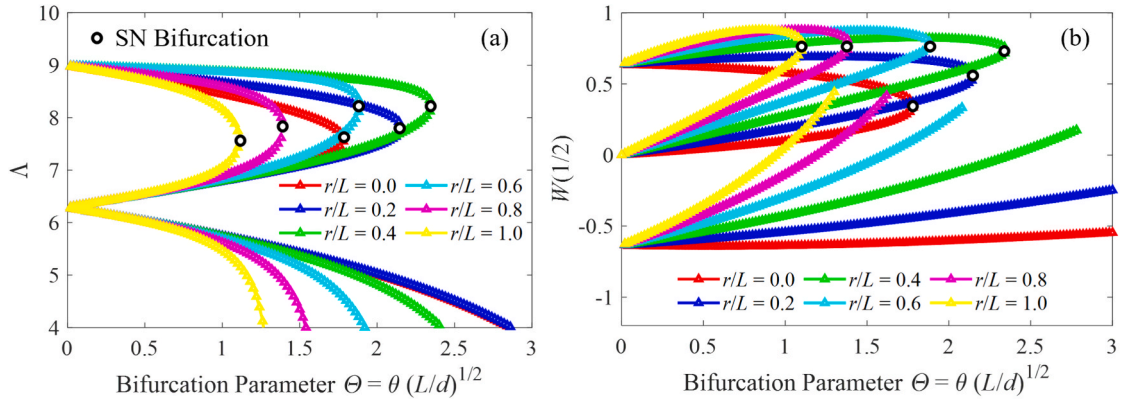


Fig. 9. Bifurcation diagrams. (a) Dimensionless longitudinal compression Δ ; (b) Midpoint deflection $W(1/2)$.

through region shown in the inset. Therefore, an element size of 1 mm was adopted in the subsequent parametric simulations to balance accuracy and computational efficiency.

To validate the finite element simulations, we studied rotated bistable buckled beams with different thicknesses and compression distances. Fig. 10 compares the quasi-static test results and numerical simulations. The finite element results agree well with the experiments. The predicted snap-through positions agree closely with the experimental results. This confirms the accuracy of the simulations. See the test video in Supplementary Video S2.

4.2.3. Parametric finite element simulations

The parameter ranges were as follows: compression distance d_0 from 1 mm to 8 mm with a step of 1 mm; beam length after compression L from 20 mm to 200 mm with a step of 5 mm; rotation radius r from 20 mm to 200 mm with a step of 5 mm. In total, 10,368 cases were simulated. In the finite element model, the beam width was 10 mm, the thickness was 2 mm, and the Young's modulus was 1.8 GPa. The subsequent data processing is independent of the cross-section and material properties. Therefore, these parameters have negligible influence on the results. The computational platform used an AMD EPYC 9654 CPU with 384 GB of RAM, enabling efficient execution of the large-scale simulations.

Fig. 11 shows a subset of the simulation results. The results were processed in the same manner as Eq. (12) to ensure independence from material and cross-sectional properties. It can be observed that rotated bistable buckled beams exhibit two snap-through events on either side of the zero angle during loading. The primary cause of this snap-through instability, as analysed in Figs. 10 and 11, is the saddle-node bifurcation, which results in a pronounced hysteresis phenomenon.

Next, the influence of structural parameters on the response of the unit is analysed. As the compression distance increases [Fig. 11 (a)], the force–displacement curve scales proportionally about the center point, preserving the negative-stiffness characteristic while expanding the negative-stiffness range. In Fig. 11(b) and 11(c), the compression distance and rotation radius are kept constant while the beam length is varied. The difference between the two figures is that in Fig. 11(b) the rotation radius is larger than the beam length, whereas in Fig. 11(c) it is smaller. It is found that when the rotation radius exceeds the beam length, the system transitions from a negative-stiffness response to a mechanical response similar to that of a “hysteron” [63], which is not suitable for the desired negative-

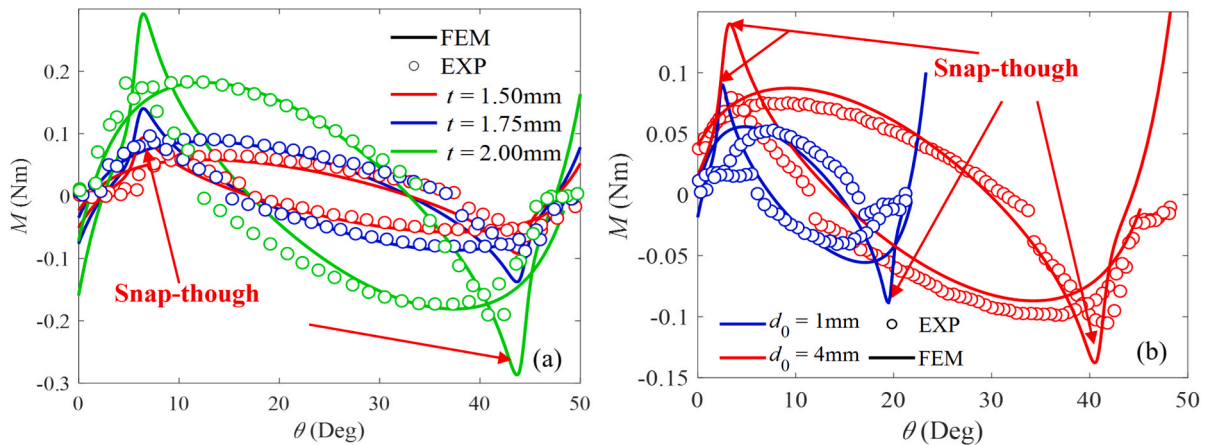


Fig. 10. Comparison of experimental and numerical results for rotated bistable buckled beams. (a) Different thicknesses; (b) Different compression distances.

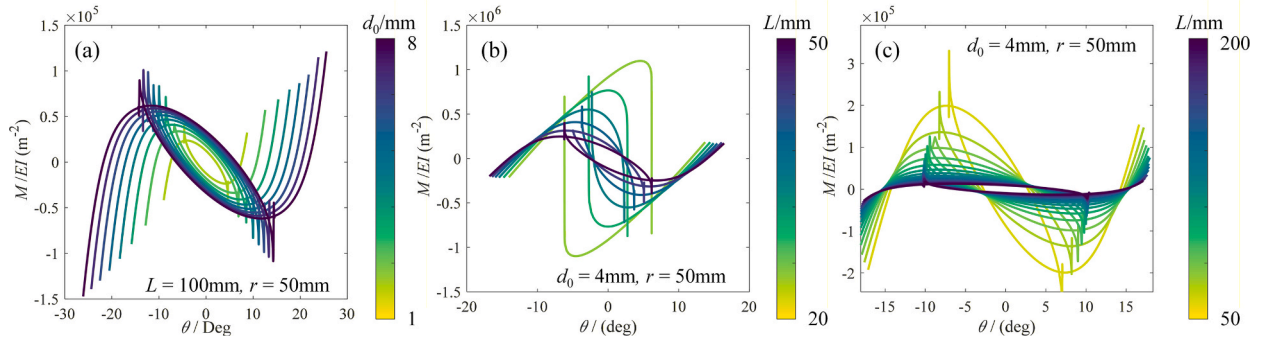


Fig. 11. Selected finite element simulation results. (a) Varying compression distance; (b) and (c) Varying beam length.

stiffness unit. When the rotation radius is smaller than the beam length, the beam exhibits a negative-stiffness region in all cases. Moreover, as L increases, the magnitude of the negative stiffness gradually decreases, and as r/L decreases, the distance between bifurcation points increases, which is consistent with the theoretical analysis. Therefore, for a fixed rotation radius, a longer beam yields a smaller negative-stiffness response but a wider negative-stiffness range.

By summarizing all the simulation data, it is found that the unit can also exhibit the response shown in Fig. 12. When the rotation radius is much larger than the beam length, the unit exhibits only a positive-stiffness response, behaving like a nonlinear positive-stiffness spring and lacking the desired negative-stiffness characteristic. Fig. 12 summarizes three typical mechanical characteristics of the rotated bistable buckled beams.

After processing all the simulation data, the unit type, the negative-stiffness magnitude K_N , and the negative-stiffness range θ_N of all beams are summarized in Fig. 13. The definitions of these parameters are given in Fig. 12. The color of the unit type corresponds to the curve color. The negative-stiffness range is the distance between the two snap-through events, and the negative-stiffness magnitude is taken as the tangential stiffness at the zero point. When $L/r > 1.55$, the system exhibits negative stiffness. The closer this ratio is to the critical value, the larger the negative stiffness but the smaller the negative-stiffness range [Fig. 13(a)–(b)]. Increasing the compression distance does not affect the overall trend of the system nor alter the fundamental mechanical behaviour of the unit, which is consistent with the conclusion drawn from Fig. 11(a).

4.3. Construction and preload adjustment of the CT-QZS module

After the stiffness characteristics of the PS and NS components have been obtained, the CT-QZS response can be constructed by connecting the two components in parallel. This subsection explains how the stiffness compensation produces the QZS characteristic

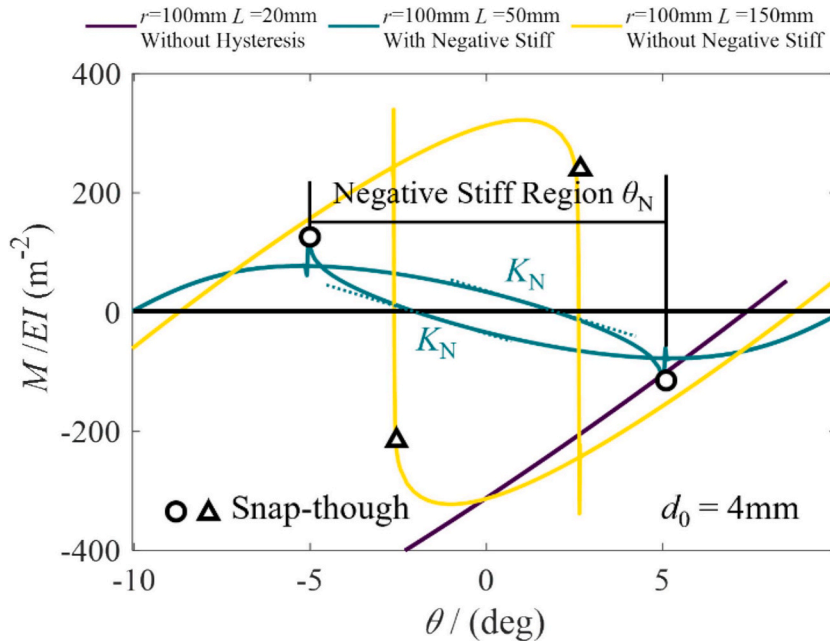


Fig. 12. Three typical mechanical characteristics of rotated bistable buckled beams.

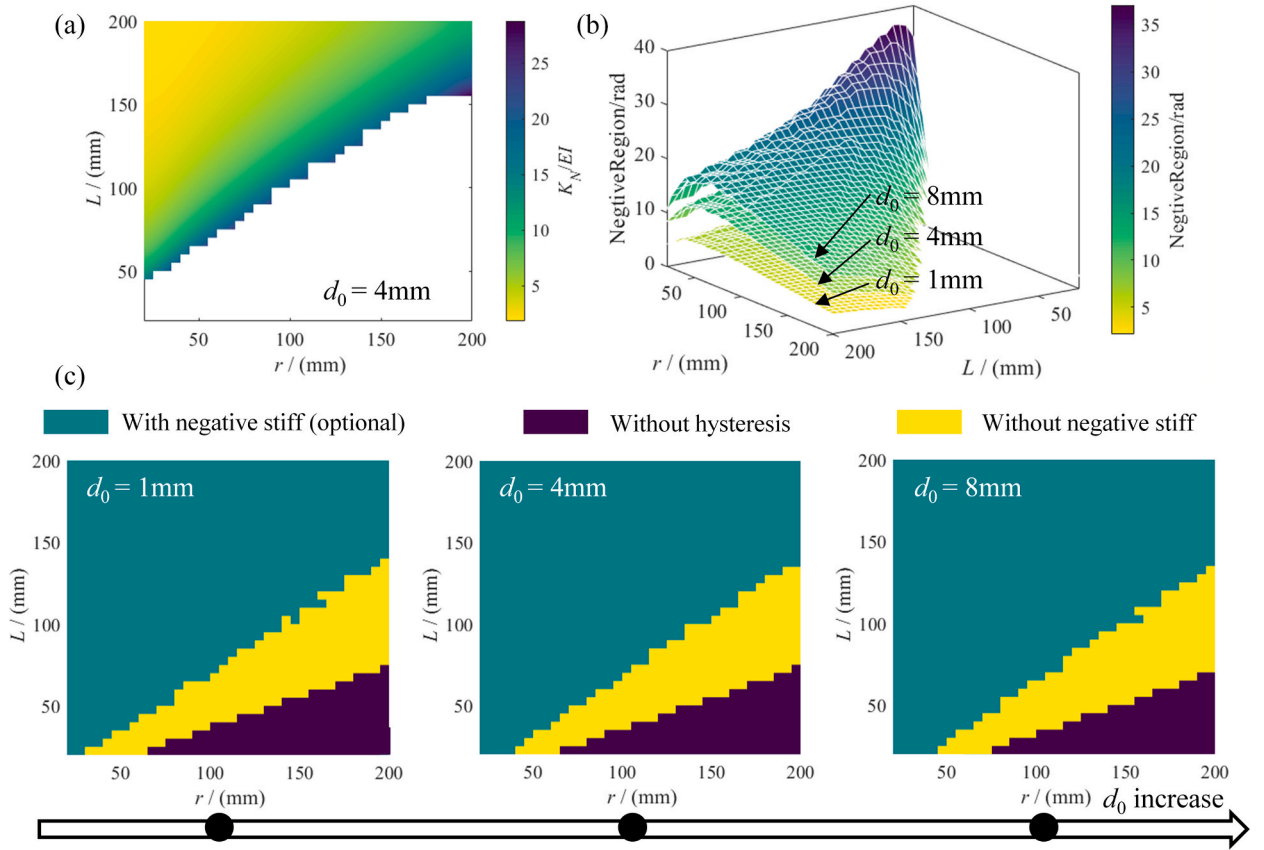


Fig. 13. Mechanical characteristic maps of rotated bistable buckled beams. (a) Negative-stiffness characteristic independent of cross-section and material; (b) Variation of negative-stiffness range with compression distance; (c) Unit characteristics with increasing compression distance.

and how the relative angular offset between the PS and NS components enables preload adjustment.

Based on the above analysis, connecting the positive and negative stiffness components in parallel yields the torsional QZS module shown in Fig. 14. Fig. 14(a) shows that the adjustable outer ring of the NS component is equipped with tunable sliding grooves, allowing an initial offset angle θ_0 between the zero points of the two modules. By adjusting this relative angular offset, the torque level

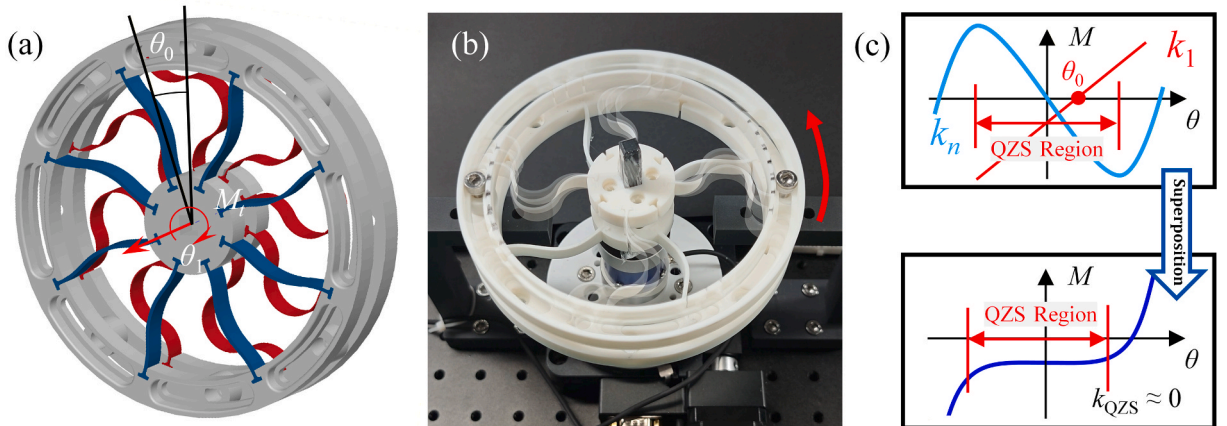


Fig. 14. Compliant torsional quasi-zero stiffness module. (a) Mechanism design for adjusting the relative offset angle θ_0 between the equilibrium points of the PS and NS modules; (b) Photograph of the prototype, where the PS component has a certain preload capacity; (c) Schematic diagram of the parallel stiffness superposition of the PS and NS modules.

of the QZS operating range can be shifted, thereby tuning the preload of the module [Fig. 14(b)]. This is the most significant difference from linear isolators, which require very soft elastic supports to lower the natural frequency and achieve low-frequency isolation. In the proposed system, however, the static preload capacity and the dynamic QZS behaviour can be adjusted more independently because the preload is tuned by θ_0 , rather than by reducing the positive stiffness of the support.

The key feature of the proposed design is that the preload is adjusted by shifting the relative equilibrium positions of the PS and NS components instead of changing their stiffnesses. Once an appropriate θ_0 is determined, superimposing the positive and negative stiffness modules yields the zero-stiffness characteristic shown in Fig. 15(c). The torque–angle relationship $M_S - \theta$ in the QZS range after superposition can be approximately expressed as

$$M_S = K_{NS}\theta + K_{PS}(\theta + \theta_0) = (K_{NS} + K_{PS})\theta + K_{PS}\theta_0 \quad (21)$$

where K_{NS} and K_{PS} are the overall stiffnesses of the NS and PS components, respectively. According to the QZS condition, $K_{NS} + K_{PS} \approx 0$ must be satisfied. Under this condition, the term $K_{PS}\theta_0$ shifts the preload torque of the QZS region, whereas the incremental torsional stiffness remains close to zero [64]. Therefore, the preload capacity can be adjusted without substantially changing the QZS operating range.

In practice, the relative offset angle is adjusted by loosening the fastening screws, rotating the adjustable outer ring to the prescribed θ_0 , and tightening the screws again to lock the ring through friction. This operation changes the preload torque of the assembled CT-QZS module without replacing the compliant beams. Fig. 15 presents the experimental results of the above system. The structural parameters of the module are listed in Table 2. The experimental video is provided in Supplementary Video S3. From Fig. 15 (a), it can be observed that as θ_0 changes, the preload of the module varies while the QZS characteristic is maintained over the same angle range. To verify the effectiveness of the system for flexible beam structures, different preloads were applied to the flexible beam structure [Fig. 15(b)]. The beam was adjusted to the desired horizontal position by tuning θ_0 , as verified using a spirit level. The flexible beam structure was made of PLA.

From Table 2, it can be seen that the geometric dimensions of the beams used in the PS and NS components are identical. Based on this, another strategy for constructing the torsional QZS characteristic is proposed in this paper, referred to as Strategy 1 and Strategy 2, as shown in Fig. 16. Since changing θ_0 mainly shifts the preload torque and does not substantially change the QZS operating range, the construction strategy first determines the required QZS range and then selects appropriate NS units to provide the required negative stiffness. The PS units are subsequently selected to compensate for the negative stiffness and satisfy the QZS condition.

If identical PS units are used, the quasi-zero-stiffness condition of the module becomes

$$K_{NS} + K_{PS} = N_P K_P + N_N K_N \approx 0 \quad (22)$$

where N_P and N_N are the numbers of PS and NS units, respectively, and K_P and K_N are the stiffnesses of a single PS and NS unit, respectively. This strategy is simple and modular, but the stiffness can only be adjusted in discrete increments because identical PS units are repeatedly used. As a result, a residual stiffness mismatch may remain in the QZS region.

$$K_{NS} + K_{PS} = \sum_{i=1}^k N_{Pi} K_{Pi} + N_N K_N \approx 0 \quad (23)$$

where K_{Pi} is the stiffness of the i -th type of PS unit, N_{Pi} is the corresponding number of units, and k is the number of PS unit types. Compared with Strategy 1, Strategy 2 provides greater flexibility in stiffness matching and can reduce the residual stiffness in the QZS region. In this study, the final CT-QZS configurations were obtained through manual and iterative experimental tuning, with the stiffness in the QZS region minimized as much as possible. The advantages of Strategy 2 are demonstrated in the subsequent vibration isolation experiments.

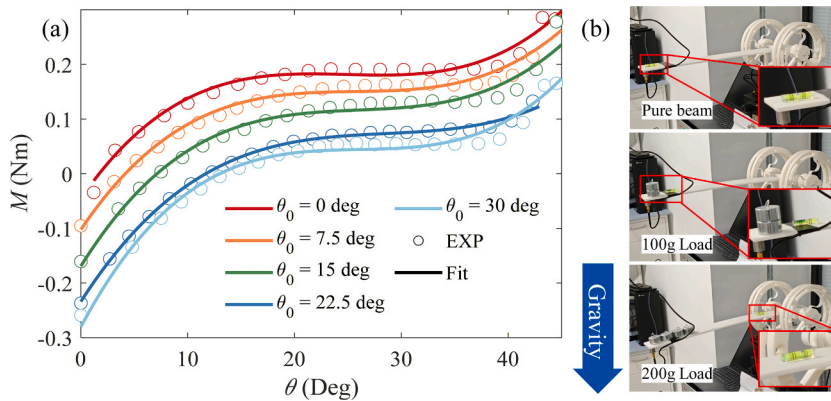
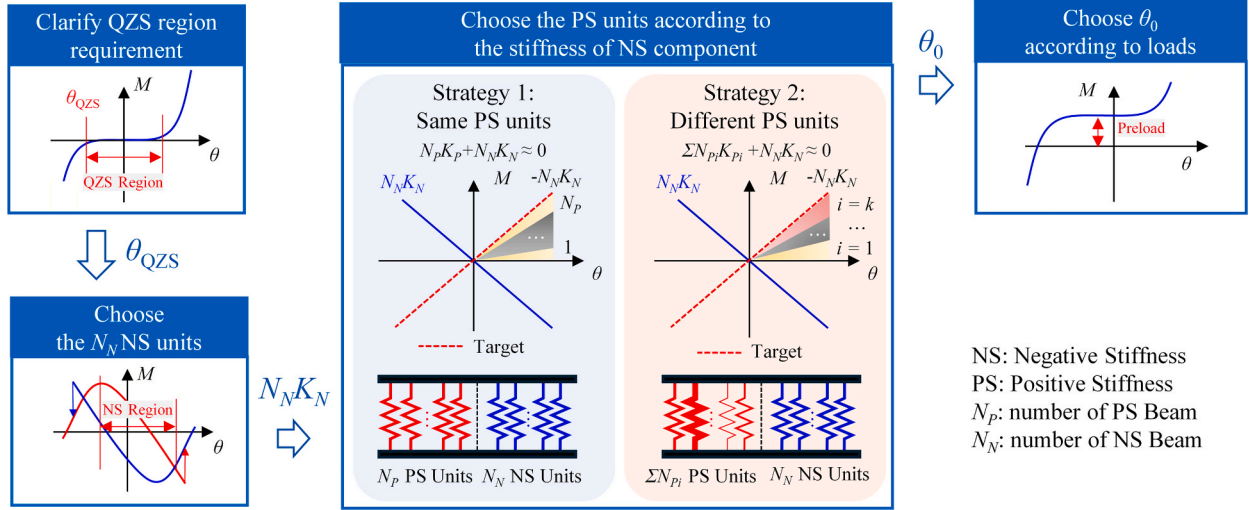


Fig. 15. Experimental results of the torsional QZS module. (a) Experimentally achieved preload-adjustable QZS characteristic; (b) Demonstration of the module applied to a flexible beam structure under different loads.

Table 2

Structural parameters of the CT-QZS module and flexible beam in Fig. 15.

Component	Parameters	Value	Unit
Both	Length L	43	mm
	Rotation radius r	20	mm
Positive Stiffness	Thickness t	2	mm
	Number of beams	8	1
Negative Stiffness	Thickness t	2	mm
	Compressed distance d_0	4	mm
	Number of beams	4	1
Flexible beam	Length l	400	mm
	Width b	50	mm
	Thickness t	4	mm

**Fig. 16.** Two construction strategies for the CT-QZS module.

5. Experimental study on vibration isolation performance

This section evaluates the vibration isolation performance of the CT-QZS isolator when integrated with a flexible beam. The mechanism by which translational base excitation induces rotational deformation of the CT-QZS module is first clarified. The natural frequencies of beams with different rotational boundaries are then analysed and validated by modal tests. Finally, swept-sine and random vibration experiments are conducted to compare the CT-QZS boundaries with the fixed boundary and linear torsional isolators.

5.1. Mechanism of base-excited rotational boundary deformation

This subsection clarifies how the translational base excitation applied by the shaker induces torsional deformation of the CT-QZS module. In the vibration tests, the shaker does not directly apply a torque to the isolator. Instead, the translational acceleration of the base generates an equivalent inertial load along the flexible beam. This inertial load produces a tendency for the beam to bend and rotate about its root. Therefore, the torsional deformation of the CT-QZS module is induced by the beam-root rotation generated under base excitation.

Let $y_b(t)$ be the base displacement and $u(x, t)$ be the transverse displacement of the beam relative to the moving base. The absolute transverse displacement of the beam $y(x, t)$ can be written as

$$y(x, t) = y_b(t) + u(x, t) \quad (24)$$

Under base excitation, the base acceleration produces an equivalent distributed inertial load along the beam

$$q(x, t) = -\rho A \ddot{y}_b(t) \quad (25)$$

where ρ and A are the density and cross-sectional area of the beam, respectively. For a uniform beam with length L_b , the corresponding generalized moment of this inertial load about the beam root can be written as

$$Q_r(x, t) = \int_0^{L_b} q(x, t) x dx = -\frac{1}{2} \rho A L_b^2 \ddot{y}_b(t) \quad (26)$$

This generalized moment is not an externally applied torque from the shaker, but an equivalent moment generated by the distributed inertial load. It is the driving source that tends to produce both beam-root rotation and elastic bending deformation.

For a fixed boundary, the beam-root rotation is constrained. As a result, the base-induced inertial effect is mainly accommodated by elastic bending deformation of the beam. This leads to relatively large curvature and bending moment near the root. In contrast, when a rotational boundary is introduced at the beam root, part of the inertial response can be accommodated by root rotation. The relative displacement of the beam can then be conceptually decomposed into a root-rotation component and an elastic bending component

$$u(x, t) \approx x\theta_r(t) + w(x, t) \quad (27)$$

where $\theta_r(t)$ is the beam-root rotation and $w(x, t)$ is the elastic deformation relative to the rotating root. The first term represents an approximate rigid-body rotation of the beam about the root and does not contribute to bending curvature. The second term determines the elastic bending deformation and the associated bending moment

$$M_b(x, t) = EI \frac{\partial^2 w}{\partial x^2} \quad (28)$$

Therefore, releasing the rotational constraint at the root changes how the base-induced inertial moment is accommodated. A considerable part of the response can be converted into beam-root rotation, while only the remaining part contributes to elastic bending deformation. This mechanism reduces the curvature and dynamic bending moment near the root compared with the fixed boundary.

The difference between a linear rotational isolator and the CT-QZS isolator lies in the root moment–rotation relationship. For a linear rotational isolator, the transmitted root moment increases approximately proportionally with the root rotation,

$$M_b(x, t) = K_\theta \theta_r(t) \quad (29)$$

where K_θ is the linear rotational stiffness. Therefore, although a linear rotational isolator can relax the root constraint and allow beam-root rotation, a larger vibration-induced root rotation still leads to an increase in the transmitted root moment.

For the CT-QZS isolator, the transmitted root moment follows the nonlinear torque–angle characteristic of the assembled module. Around the QZS operating point θ_{op} , the local incremental rotational stiffness can be defined as

$$K_{eff} = \left. \frac{dM_s}{d\theta} \right|_{\theta=\theta_{op}} \quad (30)$$

Since the torque–angle curve is nearly flat in this region, K_{eff} is close to zero. Therefore, the dynamic increment of the transmitted root moment can be approximated as

$$\Delta M_r(t) \approx K_{eff} \Delta \theta_r(t) \approx 0 \quad (31)$$

This means that, within the QZS operating range, the CT-QZS isolator allows beam-root rotation while keeping the dynamic increment of the transmitted root moment small. In other words, the root moment remains close to the preload moment instead of increasing significantly with the vibration-induced root rotation. This is the main distinction between the CT-QZS isolator and a linear

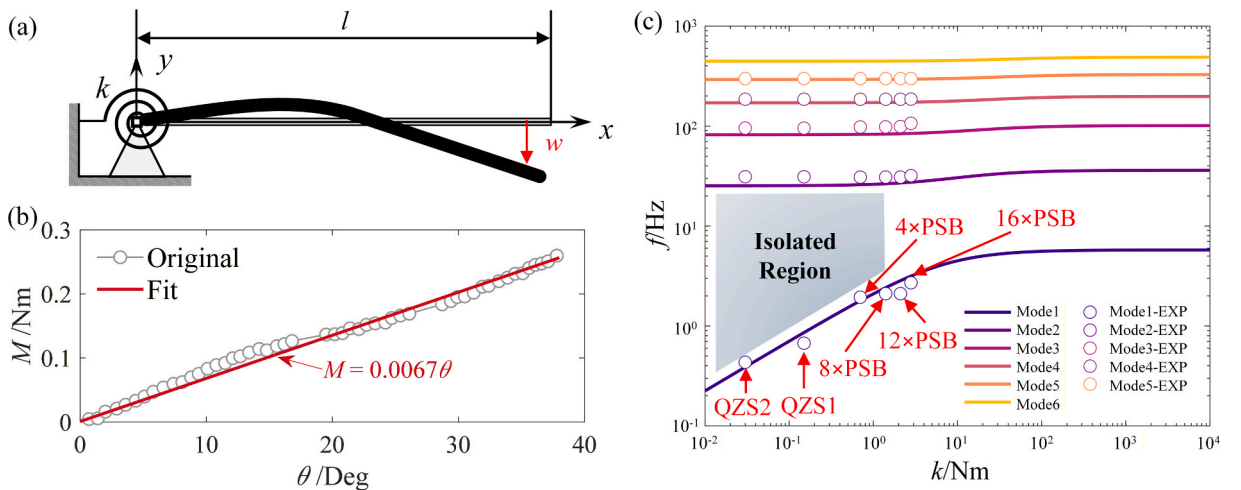


Fig. 17. Elastic boundary general beam model. (a) Schematic diagram; (b) Positive-stiffness beam used as the positive-stiffness boundary; (c) Comparison of theoretical and experimental natural frequencies of the beam under different boundary conditions.

rotational isolator.

The beam root is connected to the inner rings of the CT-QZS module through square shafts, whereas the outer rings are fixed to the support base. Consequently, the beam-root rotation produces a relative rotation between the inner and outer rings, which drives the torsional deformation of the CT-QZS module. This mechanism explains how translational base excitation is converted into torsional deformation of the CT-QZS isolator. It also explains why the proposed boundary can reduce the effective rotational constraint at the beam root, lower the first natural frequency, and improve low-frequency vibration isolation while suppressing the increase in the transmitted root moment.

5.2. Natural frequency analysis and modal validation

First, the natural frequencies of the system with a linear elastic boundary and a QZS torsional boundary were investigated. For the linear elastic boundary, it is realized by simply removing the NS module of the TQZS module. The boundary elasticity is varied by increasing or decreasing the number of positive-stiffness beams (PSBs).

First, the theoretical expression is derived. For the elastic boundary general beam model shown in Fig. 17(a), the dynamic equation

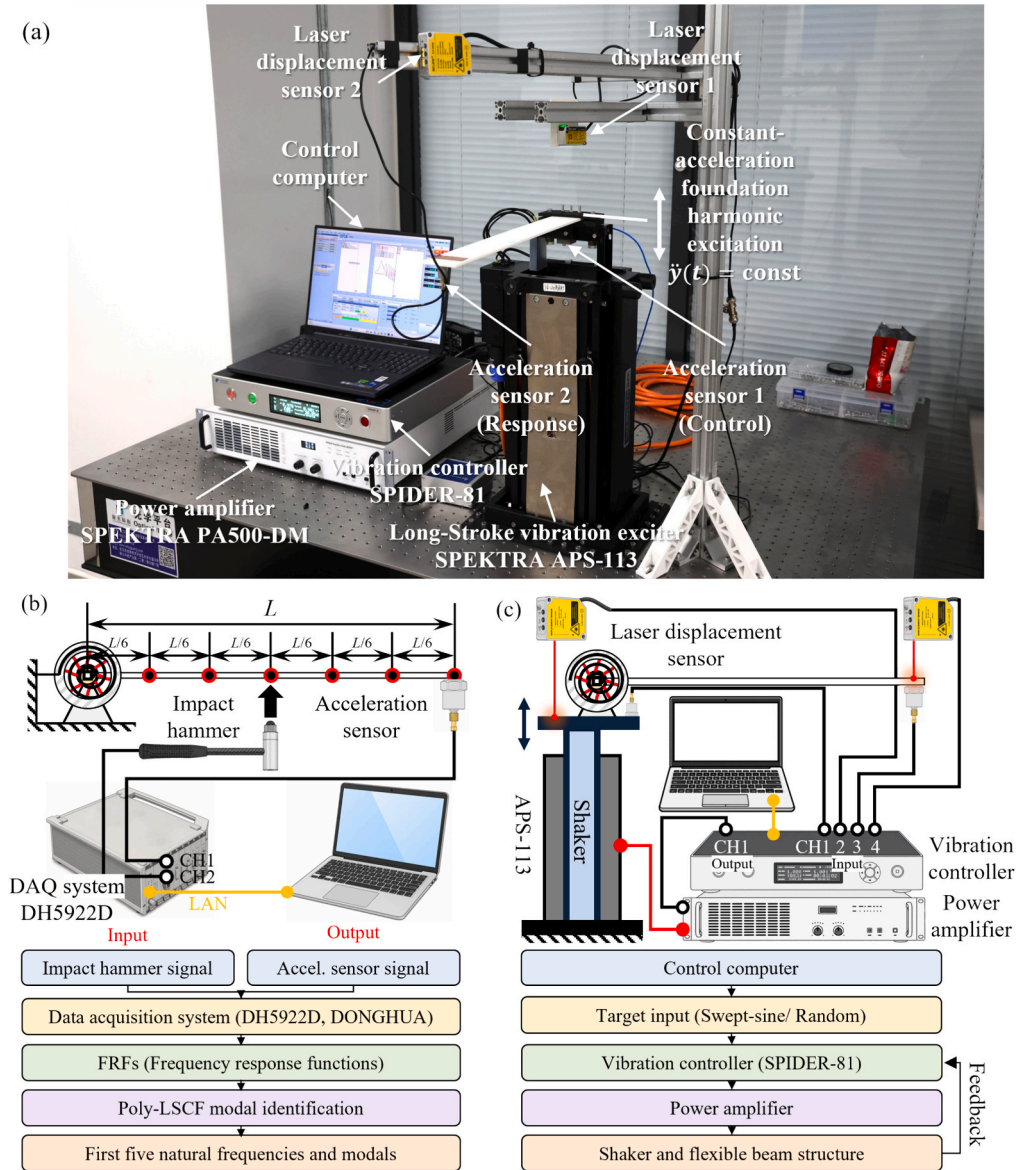


Fig. 18. Experimental setup and signal flow for modal and base-excitation vibration tests. (a) Photograph of the base-excitation test platform. (b) Modal test arrangement and Poly-LSCF identification flow. (c) Connection and closed-loop control scheme for swept-sine and random vibration tests.

of the Euler beam in the frequency domain is

$$EI \frac{\partial^4 w}{\partial x^4} - \rho A \omega^2 w = 0 \quad (32)$$

where ω is the circular frequency.

Equation (32) has the general solution of the form

$$w(x) = \bar{A} \cos(\lambda x) + \bar{B} \sin(\lambda x) + \bar{C} \cosh(\lambda x) + \bar{D} \sinh(\lambda x) \quad (33)$$

where $\bar{A}, \bar{B}, \bar{C}, \bar{D}$ are determined by the boundary conditions, and $\lambda^4 = \rho A \omega^2 / EI$. For the elastic boundary general beam, the boundary conditions are

$$k w'(0) + E I w''(0) = 0, w(0) = 0$$

$$w''(l) = 0, w'''(l) = 0 \quad (34)$$

Substituting Eq. (33) into Eq. (34) yields a system of linear equations for the unknown coefficients. By solving the determinant of the coefficient matrix, the characteristic equation of the elastic boundary general beam is obtained as

$$1 + \cos(\lambda l) \cosh(\lambda l) + \frac{EI}{k} \lambda [\cosh(\lambda l) \sin(\lambda l) - \sinh(\lambda l) \cos(\lambda l)] = 0 \quad (35)$$

Equation (35) has an infinite number of solutions. Sorted in ascending order, the i -th root λ_i corresponds to the i -th natural frequency. The i -th natural circular frequency can be written as

$$\omega_i = \sqrt{EI / \rho A l^4} (\lambda_i l)^2 \quad (36)$$

To validate the theoretical modal predictions, an impact modal test was conducted using the experimental arrangement and signal-processing procedure shown in Fig. 18(b). The modal test was performed using the impact hammer (LC-01A, SHIAO) method. Six impact points were equally spaced along the length of the beam. The structural response was measured by an accelerometer (SAED0005B, SHIAO) placed at the free end. The hammer and accelerometer signals were synchronously acquired by a data acquisition system (DH5922D, DONGHUA). The modal identification algorithm was Poly-LSCF, with a frequency range of 0.1–200 Hz, and the first five experimental modes were extracted.

Fig. 17(b) shows the quasi-static experimental results of the PS beams used as a linear elastic boundary. Fig. 17(c) compares the theoretical natural frequencies, obtained by substituting the quasi-static stiffness into Eq. (36), with the experimental results. It can be

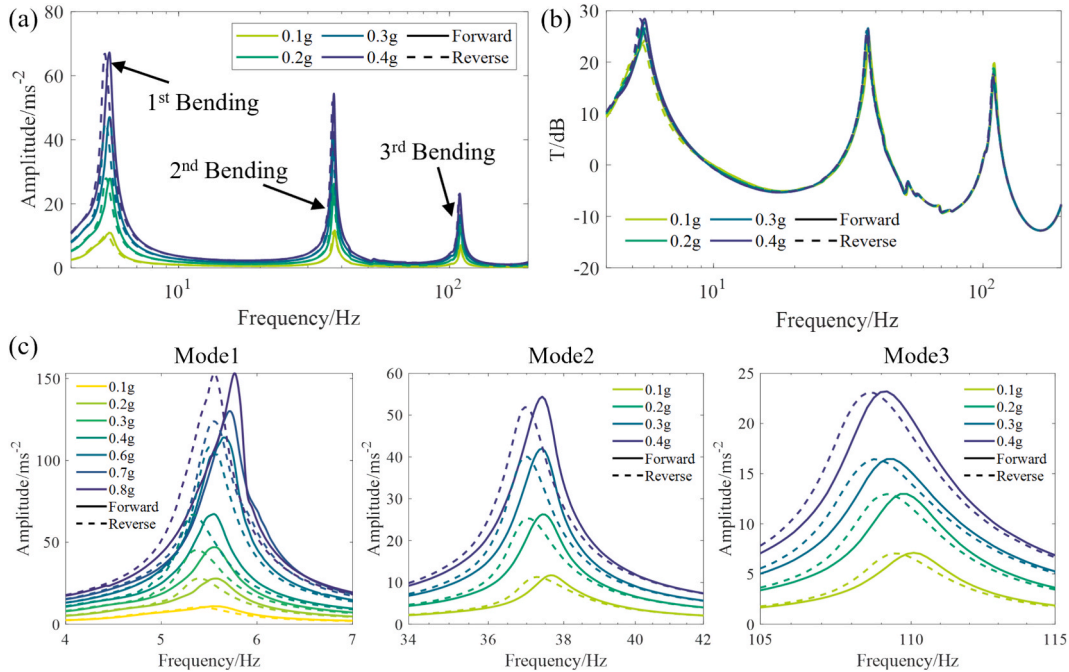


Fig. 19. Forward and backward swept-sine test results of the cantilever beam. (a) Acceleration response spectra. (b) Transmissibility curves. (c) Enlarged frequency-response curves near the first three bending modes, showing hardening behaviour in the first mode and softening behaviour in the second and third modes.

seen that the first five natural frequencies agree well. In Fig. 17(c), the number before “PSB” indicates the number of PS beams. A larger number corresponds to a stiffer boundary. QZS1 and QZS2 represent the quasi-zero-stiffness modules constructed using the two strategies in Fig. 16. The number after “QZS” indicates the corresponding strategy. It can be observed that Strategy 2 yields a lower first natural frequency. This is because its QZS range exhibits stiffness closer to zero. By lowering the natural frequency, the system achieves low dynamic stiffness for small amplitudes while maintaining high structural stiffness over a wide range, thus preserving load capacity. This strategy significantly reduces the first natural frequency without noticeably changing the higher-order natural frequencies. As a result, a wider low-frequency isolation region is obtained [Fig. 17(c)].

5.3. Vibration experiment setup for flexible beam structures

Vibration tests were conducted using a long-stroke low-frequency shaker (APS-113, SPEKTRA). The shaker has excellent low-frequency performance and is well suited for measuring low-frequency vibration characteristics of structures. A control computer was used to operate the vibration controller. It maintained a constant excitation acceleration amplitude. The shaker was equipped with a high-precision charge-type accelerometer. The signal was conditioned by a charge amplifier. It was then fed back to the control loop as an analog voltage signal. This enabled closed-loop control and constant-acceleration harmonic excitation at different frequencies. To synchronously measure the structural response, an accelerometer (SAED0005B, SHIAO) was installed at the free end of the beam to record acceleration. Two high-precision laser displacement sensors (BLG-250NM, BOJKE) were suspended from an independent supporting frame and aligned with the base-side and free-end target points of the beam. They were used to capture the time-domain response for subsequent random vibration tests. The overall vibration test platform is shown in Fig. 18(a), while the signal generation, acquisition and closed-loop control scheme for the swept-sine and random vibration tests is summarised in Fig. 18(c).

5.4. Comparative experimental analysis of CT-QZS and linear stiffness vibration isolators

To evaluate the vibration isolation performance of the proposed CT-QZS isolator as a support for the flexible beam structure, swept-sine tests were conducted in the frequency range of 3.5–200 Hz. Random vibration tests were also performed with a uniform acceleration PSD of $1 \text{ (m/s}^2\text{)}^2$ in the range of 3.5–50 Hz. The performance was compared in both the frequency and time domains.

Fig. 19 shows the swept-sine experimental results for the cantilever beam [Fig. 18(c)]. The first three bending modes exhibit clear resonance peaks within the tested frequency range [Fig. 19(a)]. After normalization by the excitation input, the transmissibility in Fig. 19(b) shows no obvious nonlinear behaviour, except for slight hardening nonlinearity near the first natural frequency. The swept-sine results of the bare cantilever beam are used as a benchmark for evaluating the isolators. Fig. 19(c) further shows the enlarged frequency–response curves in the vicinity of the first three bending resonances. A slight mismatch between the forward and reverse swept-sine responses can be observed. This mismatch is likely associated with finite sweep-rate effects, because an excessively high sweep rate may prevent the measured response from fully approximating successive steady-state states during a sine-sweep test [65]. Nevertheless, the enlarged curves reveal amplitude-dependent frequency shifts of the resonance peaks. For the first bending mode, the resonance peak shifts toward a higher frequency as the excitation level increases, indicating a hardening-type response. In contrast, the resonance peaks of the second and third bending modes shift toward lower frequencies, indicating softening-type responses.

Fig. 21 shows the enlarged swept-sine responses of QZS1 and QZS2 near the second, third and fourth bending modes. The first bending mode is not included in this figure because its natural frequency is shifted below the present swept-sine frequency range after introducing the CT-QZS isolator. For both QZS1 and QZS2, the second bending mode exhibits a softening-type nonlinear response. Around the third bending mode, the response spectra become more complicated, with multiple local peaks and less regular frequency–response profiles. For the fourth bending mode, a more pronounced softening-type response is observed, especially at higher excitation levels. These results further indicate that the beam with CT-QZS boundaries exhibits amplitude-dependent nonlinear frequency–response behaviour in the higher-mode frequency ranges.

Figs. 20 and 22 present the experimental results and key performance metrics of different isolators. It can be observed that all isolated cantilever structures, whether QZS or linear, show some degree of low-frequency isolation (Fig. 20). The advantages of isolation are not clearly visible from the frequency-domain results alone. Two key indicators are defined in Fig. 20(c): the vibration isolation onset frequency and the maximum resonance peak. The metrics of all isolators are summarized in Fig. 22. Softer linear isolators provide better low-frequency isolation. QZS isolators with near-zero stiffness show superior performance. Overall, the QZS isolator outperforms multiple linear isolators. Although the QZS isolator has a lower isolation onset frequency, the maximum resonance peak of the QZS1 isolator is significantly higher than that of the PSB16 isolator. This is attributed to the fact that PSB16 and QZS2 use more flexible beams, which introduce higher structural damping and suppress high-frequency vibrations [23]. However, damping in the boundary is not considered in the current model. This will be addressed in future work.

The best-performing QZS2 isolator and the PSB8 linear isolator were selected for random vibration tests. They were compared with the bare cantilever beam. Fig. 23(a) and (b) show the input and output power spectral densities in the frequency domain for the three cases. Fig. 23(c) shows the time-domain displacement response at the free end. The RMS (root mean square) value of the response is used to quantify the vibration level under random excitation. The input and output RMS values for the three structures are listed in Table 3. The results show that the QZS isolator performs better under the same random vibration conditions. Supplementary Video S4 shows the responses of the three configurations under random excitation. Although the QZS isolator does not show a dramatic improvement over the linear isolator, its low-dynamic-stiffness and high-static-stiffness characteristics, together with the adjustable preload mechanism proposed in this work, provide higher load capacity and structural stiffness compared with linear isolators.

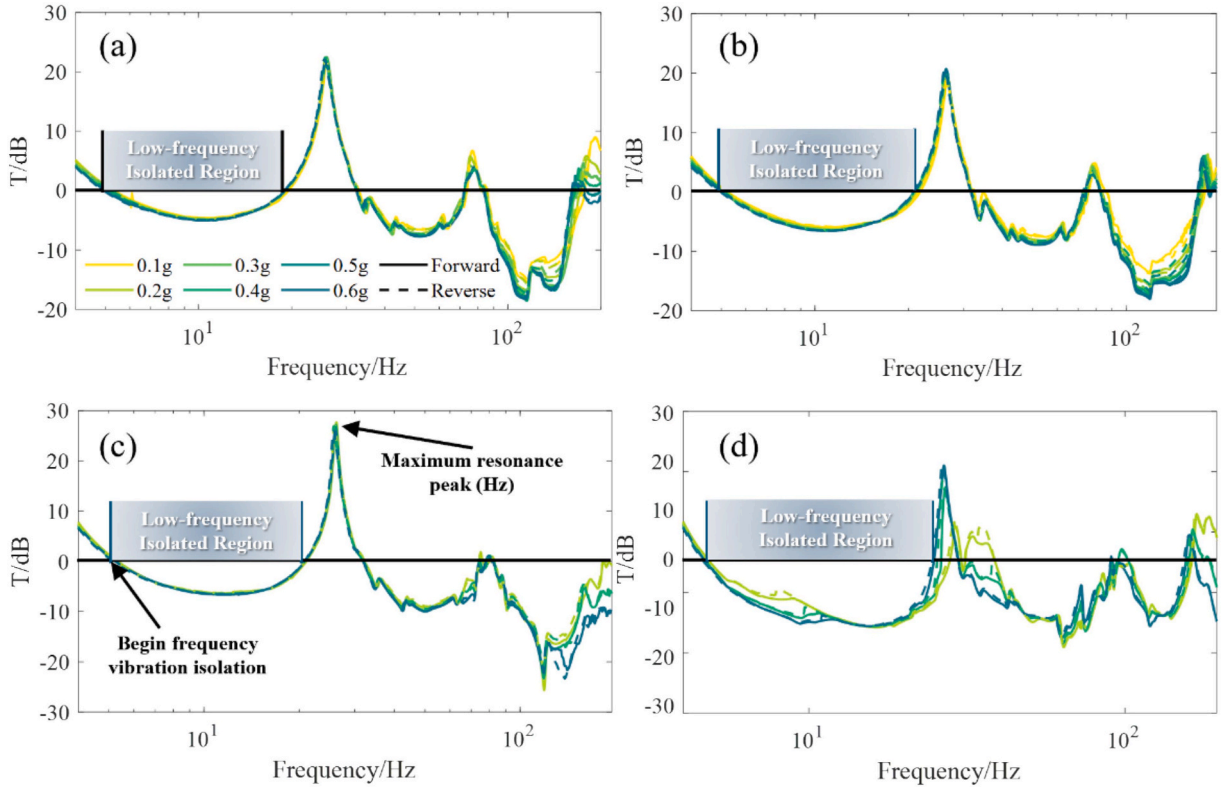


Fig. 20. Transmissibility–frequency response of the cantilever structure with different isolators. (a) QZS1; (b) QZS2; (c) $12 \times$ PSB; (d) $16 \times$ PSB isolator.

6. Discussion

6.1. Potential engineering application

The proposed CT-QZS isolators can be applied to deployable satellite solar-array structures, as illustrated in Fig. 24. In such systems, solar panels are folded against the satellite bus during launch and then deployed through the root hinge and panel-to-panel hinges after release. The CT-QZS module could be integrated into these hinge interfaces to provide both deployment torque and post-deployment vibration isolation.

During deployment, the recovered stiffness outside the QZS region enables the CT-QZS hinge to generate a restoring torque, so the module can also act as a passive driving element for panel rotation. The fully deployed configuration is then designed to fall within the QZS operating region. Specifically, the root CT-QZS hinge may be designed to operate around the deployed equilibrium angle of approximately 90° , while the panel-to-panel CT-QZS hinges may be designed to operate around approximately 180° , as shown schematically in Fig. 24(b,c).

After deployment and locking, the solar array behaves as a lightweight flexible appendage connected to the satellite bus through rotational interfaces. In the QZS operating region, the low dynamic torsional stiffness allows small vibration-induced hinge rotations with only small torque variations, thereby reducing low-frequency free vibration and dynamic bending-moment transmission between the solar array and the satellite bus. Outside this region, the stiffness recovers and provides load-carrying capability under larger disturbances. Therefore, the CT-QZS hinge combines three functions in this application: passive deployment assistance, deployed-state vibration isolation, and dynamic moment-transfer reduction.

6.2. Limitations

The proposed CT-QZS isolator is a torsional isolator and cannot directly isolate translational vibration by itself. For applications involving coupled translational and rotational excitations, it should be integrated with translational or multi-directional isolation modules [14].

The nonlinear dynamics of the flexible beam–CT-QZS system have not been systematically studied. The swept-sine tests show amplitude-dependent nonlinear frequency–response behaviour, but a complete analysis of nonlinear resonances, jump phenomena, bifurcations and possible chaotic responses [66] is beyond the scope of this work. Future studies should develop a coupled nonlinear

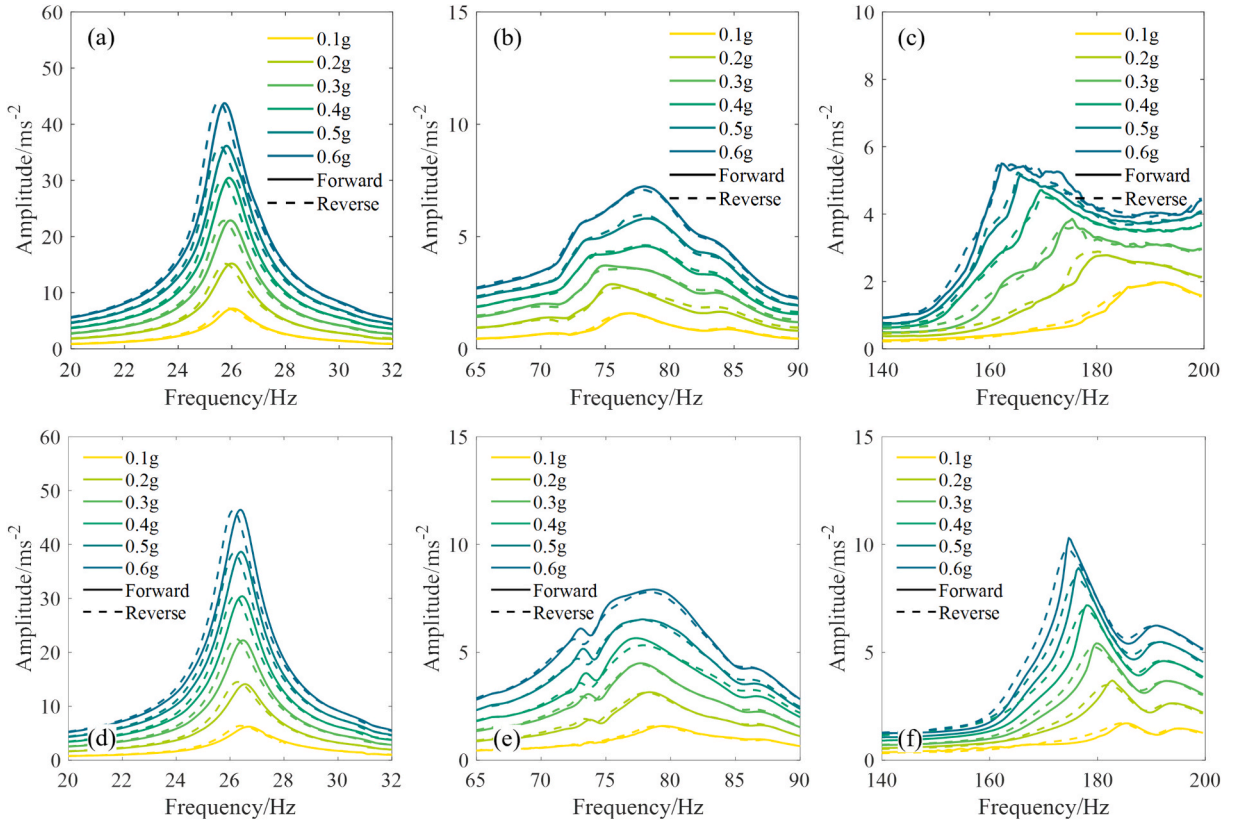


Fig. 21. Enlarged nonlinear frequency responses of the beam with CT-QZS boundaries near higher bending modes. (a–c) QZS1 and (d–f) QZS2. From left to right, the columns show the swept-sine responses near the second, third and fourth bending modes, respectively.

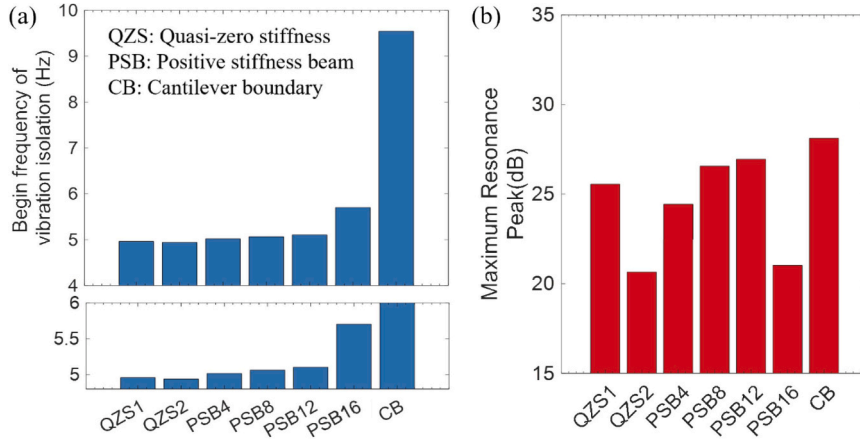


Fig. 22. Critical metrics of the flexible beam structure with different isolators: (a) Vibration isolation onset frequency; (b) Maximum resonance peak.

model including the large-deformation beam, the nonlinear torque–rotation relation of the CT-QZS module and their boundary interaction [67,68].

The TPU-based NS component was modelled mainly using an equivalent elastic modulus in the present study. However, TPU is viscoelastic, and its dynamic stiffness and energy dissipation depend on frequency, amplitude and loading history. Therefore, the relative contributions of QZS stiffness modulation, structural nonlinearity and TPU material dissipation remain to be quantified. Future work should introduce viscoelastic or fractional-derivative material models to evaluate the influence of TPU damping on the isolation performance [25].

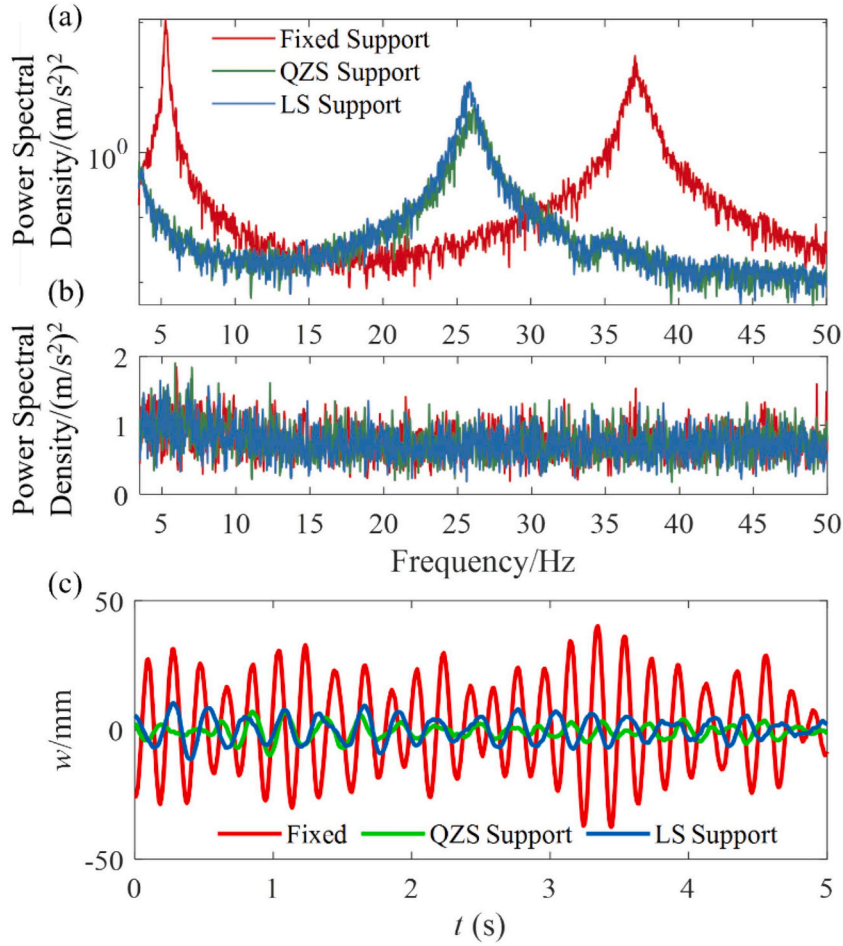


Fig. 23. Random vibration experimental results. (a) Output power spectral density at the free end; (b) Input power spectral density of the base excitation; (c) Time-domain displacement response at the free end.

Table 3
Random vibration test results.

Boundary	Input RMS(m/s ²)	Output RMS(m/s ²)
Fixed	8.8869	36.6548
QZS2 Isolator	8.7757	13.9953
PSB8 Isolator	8.6524	17.7137

7. Conclusions

This paper has proposed, modelled, and experimentally validated a preload-adjustable CT-QZS isolator for low-frequency vibration isolation of flexible beam structures. The main conclusions are as follows:

(1) A theoretical model of the PS component was established using Euler beam theory and the unit-load method. The torque–angle relationship was derived. Quasi-static experiments validated the accuracy of the model. The NS component, composed of rotated bistable buckled beams, was studied through bifurcation analysis and large-scale parametric finite element simulations. The results show that the emergence of negative stiffness depends on the ratio of beam length to rotation radius. A critical value exists, beyond which negative stiffness appears. By adjusting the compression distance and the length-to-radius ratio, both the magnitude and the width of the negative-stiffness region can be tuned. This provides clear guidance for isolator design.

(2) A key contribution of this work is the introduction of a tunable sliding groove. It allows the relative angle between the PS and NS components to be adjusted. Rotating the PS component changes the preload, while the quasi-zero-stiffness range remains unchanged. Experiments confirm that the proposed isolator maintains a constant zero-stiffness interval while allowing continuous preload adjustment. This enables adaptation to different static load conditions. In contrast, linear isolators with low stiffness do not offer preload adjustability and are limited to fixed load conditions.

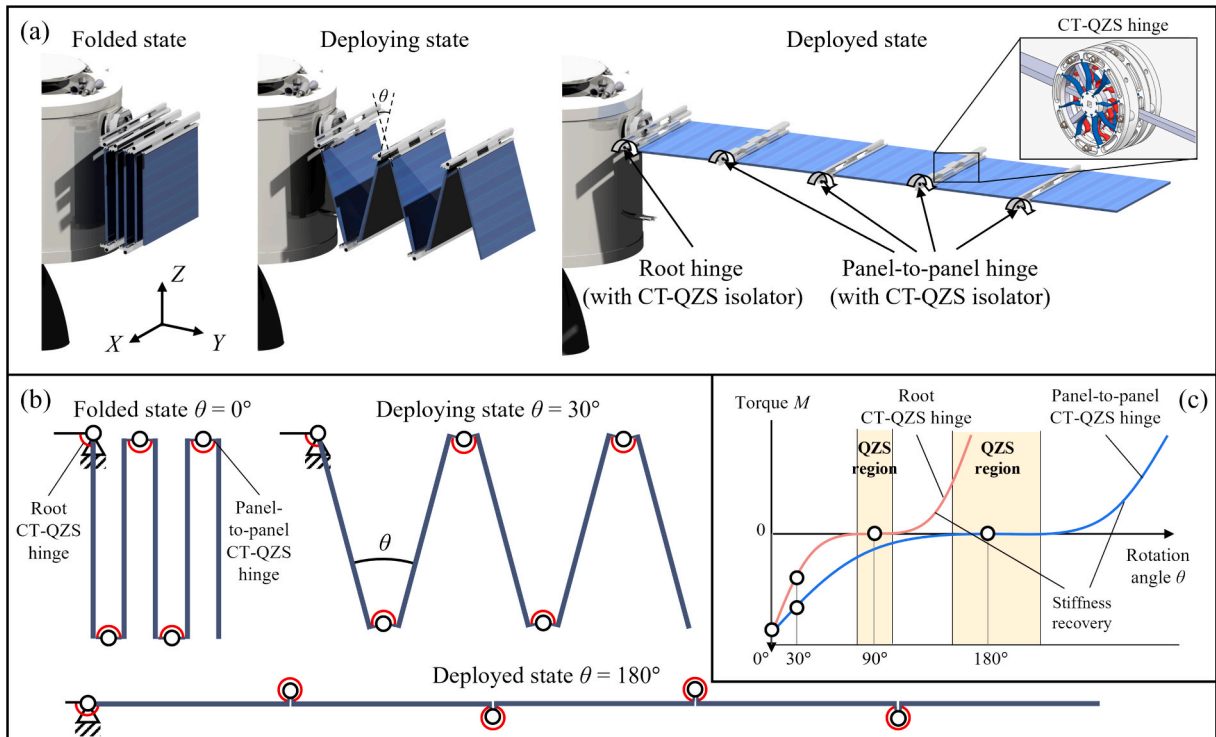


Fig. 24. Conceptual application of CT-QZS isolators in a deployable satellite solar-array structure. (a) Folded, deploying and deployed configurations. (b) Simplified hinge-chain model with CT-QZS hinges at the root and panel-to-panel interfaces. (c) Schematic torque-rotation characteristics and QZS operating ranges.

(3) Modal, swept-sine and random vibration tests demonstrated the low-frequency isolation capability of the CT-QZS isolator. Compared with the fixed boundary and linear torsional isolators, the CT-QZS isolator reduced the first natural frequency and shifted the vibration isolation onset to a lower frequency range. The isolation onset frequencies of QZS1 and QZS2 were 5.0 Hz and 4.9 Hz, respectively. Under random excitation, the best-performing CT-QZS isolator reduced the output acceleration RMS by 61.8% compared with the fixed boundary and by 21.0% compared with the PSB8 linear isolator.

Future work will introduce viscoelastic damping into the boundary model to further suppress high-frequency resonance peaks and extend the design to multi-directional (translational-rotational coupled) vibration isolation.

CRediT authorship contribution statement

Ke Huang: Writing – original draft, Validation, Investigation. **Jiaying Zhang:** Supervision, Conceptualization. **Qingyun Wang:** Writing – review & editing. **Alexander D. Shaw:** Writing – review & editing. **Michael I. Friswell:** Writing – review & editing, Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Material dependency and constitutive-model assessment of the NS component

The NS component is made of TPU 95A because the rotated bistable buckled beams undergo repeated large-deformation snap-through during operation. Since TPU exhibits hyperelastic behaviour, additional finite element simulations were conducted to examine whether the linear elastic approximation used in the theoretical and parametric analyses is acceptable within the operating

deformation range of the present design. Three representative compression distances, $d_0 = 1, 4$, and 8 mm, were considered. In all cases, the beam thickness was fixed at 1 mm and the radial distance between the outer and inner rings was kept constant at 43 mm.

The TPU material was modelled in two ways. First, an equivalent linear elastic model was used, with the elastic modulus obtained from the linear fitting of the tensile test data in the strain range below 0.1 . Second, a third-order Ogden hyperelastic model was fitted from the same material test data and implemented in ABAQUS. The torque–angle responses predicted by the two constitutive models are compared in Fig. A1(a). For all three compression distances, the linear elastic and hyperelastic models give very similar responses. Small differences appear near the large-rotation snap-through branches, but these differences do not alter the predicted negative-stiffness trend or the design range used for CT-QZS construction. This indicates that the global negative-stiffness characteristic is governed mainly by geometric nonlinearity and bistable buckling, rather than by the constitutive nonlinearity of TPU.

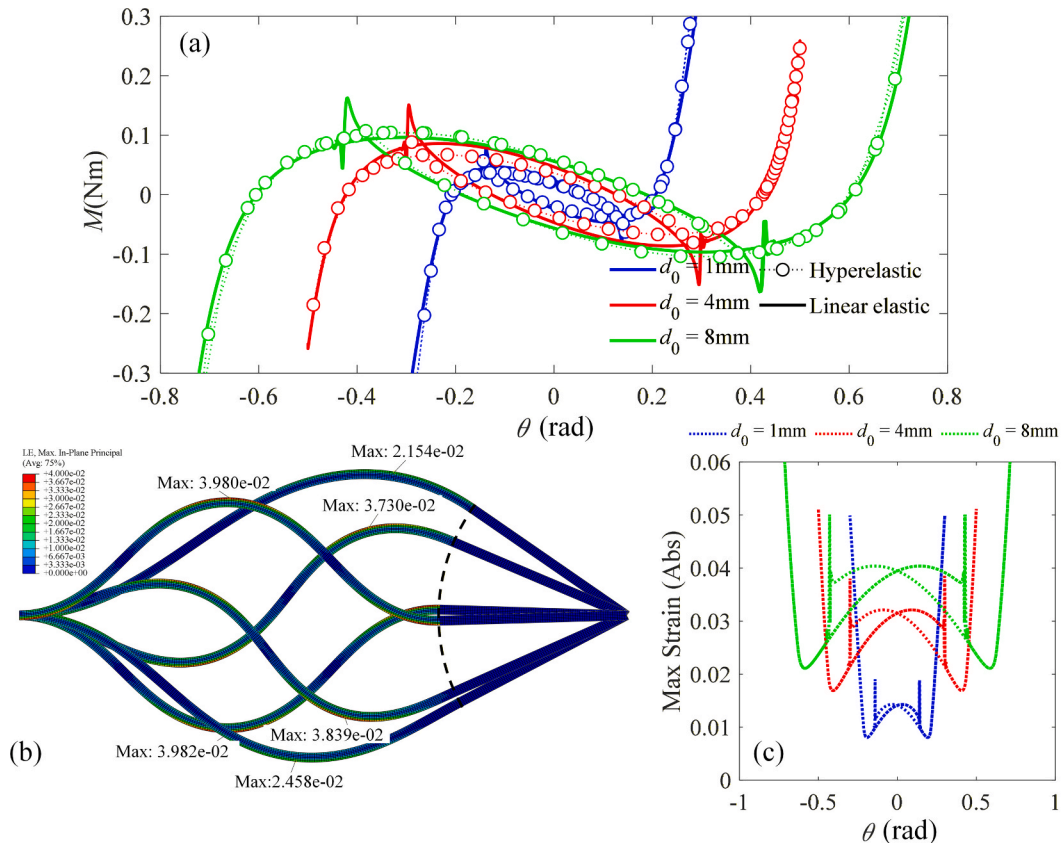


Fig. A1. Material-dependency assessment of the NS component. (a) Linear elastic and hyperelastic torque–angle responses. (b) Strain contours during snap-through deformation. (c) Maximum strain versus rotation angle.

A strain-level assessment was further conducted to clarify the material requirement of the NS component. Fig. A1(b) shows representative strain contours of the rotated bistable buckled beams, and Fig. A1(c) plots the maximum strain as a function of the rotation angle for $d_0 = 1, 4$, and 8 mm. The maximum strain increases with the compression distance. For the larger compression distance, the strain level reaches approximately (0.04) in the main operating region and increases further near the large-rotation snap-through branches. This strain level is significantly higher than the initial linear elastic strain range of the printed PLA specimens, which depart from the linear regime at approximately 0.02 strain and fracture at approximately 0.04 strain, as shown in Fig. 3(b). Therefore, PLA is not suitable for the present NS geometry under repeated snap-through deformation without geometric re-scaling.

These results confirm that the use of TPU is not a weakness of the proposed design, but a necessary material choice for achieving recoverable large-deformation bistable behaviour. Meanwhile, because the strain range experienced by the NS component remains within or close to the strain range used for the effective-modulus fitting, and because the linear and hyperelastic simulations lead to similar torque–angle responses, the linear elastic approximation is considered adequate for the theoretical derivation and parametric design presented in the main text.

Appendix B. Supplementary data

Supplementary data to this article can be found online at <https://doi.org/10.1016/j.ymsp.2026.114680>.

Data availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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